

# Actuarial Mathematics

Yosuke Ito

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## Abstract

Actuarial Mathematics is a theory in applied mathematics, which is mainly used for determining the prices of insurance products and evaluating the liability of a company associating with insurance contracts. It is related to calculus, probability theory and financial theory, etc.

In this entry, I formalize the very basic part of Actuarial Mathematics in Isabelle/HOL. It includes the theory of interest, survival model, and life table. The theory of interest deals with interest rates, present value factors, an annuity certain, etc. The survival model is a probabilistic model that represents the human mortality. The life table is based on the survival model and used for practical calculations.

I have already formalized the basic part of Actuarial Mathematics in Coq (<https://github.com/Yosuke-Ito-345/Actuary>) in a purely axiomatic manner. In contrast, Isabelle formalization is based on the probability theory and the survival model is developed as generally as possible. Such rigorous and general formulation seems very rare; at least I cannot find any similar documentation on the web.

This formalization in Isabelle is still at an early stage, and I cannot guarantee the backward compatibility in the future development. If you heavily depend on the “Actuarial Mathematics” library, please let me know.

## Contents

<b>1</b>	<b>Preliminary Lemmas</b>	<b>1</b>
1.1	Lemmas on <i>indicator</i> for a Linearly Ordered Type . . . . .	7
<b>2</b>	<b>Additional Lemmas for the <i>HOL–Analysis</i> Library</b>	<b>10</b>
2.1	Set Lebesgue Integrability on Affine Transformation . . . . .	16
2.2	Set Lebesgue Integral on Affine Transformation . . . . .	18
2.3	Alternative Integral Test . . . . .	19
2.4	Interchange of Differentiation and Lebesgue Integration . . . . .	20

<b>3</b>	<b>Additional Lemmas for the <i>HOL-Probability</i> Library</b>	<b>22</b>
3.1	More Properties of <i>cdf</i> 's . . . . .	23
3.2	Conditional Probability Space . . . . .	25
3.3	Complementary Cumulative Distribution Function . . . . .	26
3.4	Hazard Rate . . . . .	29
<b>4</b>	<b>Theory of Interest</b>	<b>31</b>
<b>5</b>	<b>Survival Model</b>	<b>38</b>
5.1	General Theory of Survival Model . . . . .	39
5.1.1	Introduction of Survival Function for $X$ . . . . .	39
5.1.2	Introduction of Future-Lifetime Random Variable $T(x)$ . . . . .	40
5.1.3	Actuarial Notations on the Survival Model . . . . .	41
5.1.4	Properties of Survival Function for $T(x)$ . . . . .	42
5.1.5	Properties of $\$p\{-t&x\}$ . . . . .	44
5.1.6	Properties of Survival Function for $X$ . . . . .	44
5.1.7	Introduction of Cumulative Distributive Function for $X$ . . . . .	45
5.1.8	Properties of Cumulative Distributive Function for $T(x)$ . . . . .	45
5.1.9	Properties of $\$q\{-t&x\}$ . . . . .	47
5.1.10	Properties of Cumulative Distributive Function for $X$ . . . . .	48
5.1.11	Relations between $\$p\{-t&x\}$ and $\$q\{-t&x\}$ . . . . .	48
5.1.12	Properties of Life Expectation . . . . .	49
5.2	Piecewise Differentiable Survival Function . . . . .	52
5.2.1	Properties of Survival Function for $X$ . . . . .	52
5.2.2	Properties of Cumulative Distributive Function for $X$ . . . . .	52
5.2.3	Introduction of Probability Density Functions of $X$ and $T(x)$ . . . . .	53
5.2.4	Properties of Survival Function for $T(x)$ . . . . .	53
5.2.5	Properties of Cumulative Distributive Function for $T(x)$ . . . . .	54
5.2.6	Properties of Probability Density Function of $T(x)$ . . . . .	55
5.2.7	Properties of Probability Density Function of $X$ . . . . .	57
5.2.8	Relations between Life Expectation and Probability Density Function . . . . .	58
5.2.9	Introduction of Force of Mortality . . . . .	59
5.2.10	Properties of Force of Mortality . . . . .	60
5.2.11	Properties of Curtate Life Expectation . . . . .	62
5.3	Limited Survival Function . . . . .	63
<b>6</b>	<b>Life Table</b>	<b>64</b>
6.1	Basic Properties of Life Table . . . . .	64
6.2	Construction of Survival Model from Life Table . . . . .	67
6.2.1	Relations between Life Table and Survival Function for $X$ . . . . .	68

6.2.2	Relations between Life Table and Cumulative Distributive Function for $X$ . . . . .	69
6.2.3	Relations between Life Table and Survival Function for $T(x)$ . . . . .	69
6.2.4	Relations between Life Table and Cumulative Distributive Function for $T(x)$ . . . . .	71
6.2.5	Life Table and Actuarial Notations . . . . .	71
6.3	Piecewise Differentiable Life Table . . . . .	74
6.4	Interpolations . . . . .	77
6.5	Limited Life Table . . . . .	81

## 7 Examples 82

**theory** *Preliminaries*

**imports** *HOL-Library.Rewrite HOL-Library.Extended-Nonnegative-Real HOL-Library.Extended-Real HOL-Probability.Probability*

**begin**

**declare** *[[show-types]]*

**notation** *powr (infixr <. > 80)*

## 1 Preliminary Lemmas

**lemma** *Collect-conj-eq2*:  $\{x \in A. P x \wedge Q x\} = \{x \in A. P x\} \cap \{x \in A. Q x\}$   
*<proof>*

**lemma** *vimage-compl-atMost*:  
**fixes**  $f :: 'a \Rightarrow 'b::linorder$   
**shows**  $-(f - \cdot \{..y\}) = f - \cdot \{y<..\}$   
*<proof>*

**context** *linorder*

**begin**

**lemma** *Icc-minus-Ico*:  
**fixes**  $a b$   
**assumes**  $a \leq b$   
**shows**  $\{a..b\} - \{a..<b\} = \{b\}$   
*<proof>*

**lemma** *Icc-minus-Ioc*:  
**fixes**  $a b$   
**assumes**  $a \leq b$   
**shows**  $\{a..b\} - \{a<..b\} = \{a\}$   
*<proof>*

**lemma** *Int-atLeastAtMost-Unbounded[simp]*:  $\{a.. \} \text{Int} \{..b\} = \{a..b\}$   
*<proof>*

**lemma** *Int-greaterThanAtMost-Unbounded[simp]*:  $\{a < ..\} \text{ Int } \{..b\} = \{a < ..b\}$   
 ⟨proof⟩

**lemma** *Int-atLeastLessThan-Unbounded[simp]*:  $\{a ..\} \text{ Int } \{..< b\} = \{a ..< b\}$   
 ⟨proof⟩

**lemma** *Int-greaterThanLessThan-Unbounded[simp]*:  $\{a < ..\} \text{ Int } \{..< b\} = \{a < ..< b\}$   
 ⟨proof⟩

**end**

**lemma** *Ico-real-nat-disjoint*:  
*disjoint-family*  $(\lambda n :: \text{nat}. \{a + \text{real } n ..< a + \text{real } n + 1\})$  **for**  $a :: \text{real}$   
 ⟨proof⟩

**corollary** *Ico-nat-disjoint*: *disjoint-family*  $(\lambda n :: \text{nat}. \{\text{real } n ..< \text{real } n + 1\})$   
 ⟨proof⟩

**lemma** *Ico-real-nat-union*:  $(\bigcup n :: \text{nat}. \{a + \text{real } n ..< a + \text{real } n + 1\}) = \{a ..\}$   
**for**  $a :: \text{real}$   
 ⟨proof⟩

**corollary** *Ico-nat-union*:  $(\bigcup n :: \text{nat}. \{\text{real } n ..< \text{real } n + 1\}) = \{0 ..\}$   
 ⟨proof⟩

**lemma** *Ico-nat-union-finite*:  $(\bigcup (n :: \text{nat}) < m. \{\text{real } n ..< \text{real } n + 1\}) = \{0 ..< m\}$   
 ⟨proof⟩

**lemma** *seq-part-multiple*: **fixes**  $m n :: \text{nat}$  **assumes**  $m \neq 0$  **defines**  $A \equiv \lambda i :: \text{nat}. \{i * m ..< (i + 1) * m\}$   
**shows**  $\forall i j. i \neq j \longrightarrow A i \cap A j = \{\}$  **and**  $(\bigcup i < n. A i) = \{..< n * m\}$   
 ⟨proof⟩

**lemma** *frontier-Icc-real*: *frontier*  $\{a .. b\} = \{a, b\}$  **if**  $a \leq b$  **for**  $a b :: \text{real}$   
 ⟨proof⟩

**lemma**(**in field**) *divide-mult-cancel[simp]*: **fixes**  $a b$  **assumes**  $b \neq 0$   
**shows**  $a / b * b = a$   
 ⟨proof⟩

**lemma** *inverse-powr*:  $(1/a) . \hat{\ } b = a . \hat{\ } -b$  **if**  $a > 0$  **for**  $a b :: \text{real}$   
 ⟨proof⟩

**lemma** *powr-eq-one-iff-gen[simp]*:  $a . \hat{\ } x = 1 \longleftrightarrow x = 0$  **if**  $a > 0$   $a \neq 1$  **for**  $a x :: \text{real}$   
 ⟨proof⟩

**lemma** *powr-less-cancel2*:  $0 < a \implies 0 < x \implies 0 < y \implies x . \hat{\ } a < y . \hat{\ } a \implies x <$

*y*

**for**  $a\ x\ y :: \text{real}$

$\langle \text{proof} \rangle$

**lemma** *geometric-increasing-sum-aux*:  $(1-r)^{\wedge}2 * (\sum k < n. (k+1)*r^{\wedge}k) = 1 - (n+1)*r^{\wedge}n + n*r^{\wedge}(n+1)$

**for**  $n :: \text{nat}$  **and**  $r :: \text{real}$

$\langle \text{proof} \rangle$

**lemma** *geometric-increasing-sum*:  $(\sum k < n. (k+1)*r^{\wedge}k) = (1 - (n+1)*r^{\wedge}n + n*r^{\wedge}(n+1)) / (1-r)^{\wedge}2$

**if**  $r \neq 1$  **for**  $n :: \text{nat}$  **and**  $r :: \text{real}$

$\langle \text{proof} \rangle$

**lemma** *Reals-UNIV[simp]*:  $\mathbb{R} = \{x :: \text{real}. \text{True}\}$

$\langle \text{proof} \rangle$

**lemma** *Lim-cong*:

**assumes**  $\forall_F x \text{ in } F. f\ x = g\ x$

**shows**  $\text{Lim } F\ f = \text{Lim } F\ g$

$\langle \text{proof} \rangle$

**lemma** *LIM-zero-iff'*:  $((\lambda x. l - f\ x) \longrightarrow 0) \iff F = (f \longrightarrow l) \iff F$

**for**  $f :: 'a \Rightarrow 'b :: \text{real-normed-vector}$

$\langle \text{proof} \rangle$

**lemma** *antimono-onI*:

$(\bigwedge r\ s. r \in A \implies s \in A \implies r \leq s \implies f\ r \geq f\ s) \implies \text{antimono-on } A\ f$

$\langle \text{proof} \rangle$

**lemma** *antimono-onD*:

$\llbracket \text{antimono-on } A\ f; r \in A; s \in A; r \leq s \rrbracket \implies f\ r \geq f\ s$

$\langle \text{proof} \rangle$

**lemma** *antimono-imp-mono-on*:  $\text{antimono } f \implies \text{antimono-on } A\ f$

$\langle \text{proof} \rangle$

**lemma** *antimono-on-subset*:  $\text{antimono-on } A\ f \implies B \subseteq A \implies \text{antimono-on } B\ f$

$\langle \text{proof} \rangle$

**lemma** *mono-on-antimono-on*:

**fixes**  $f :: 'a :: \text{order} \Rightarrow 'b :: \text{ordered-ab-group-add}$

**shows**  $\text{mono-on } A\ f \iff \text{antimono-on } A\ (\lambda r. - f\ r)$

$\langle \text{proof} \rangle$

**corollary** *mono-antimono*:

**fixes**  $f :: 'a :: \text{order} \Rightarrow 'b :: \text{ordered-ab-group-add}$

**shows**  $\text{mono } f \iff \text{antimono } (\lambda r. - f\ r)$

$\langle \text{proof} \rangle$

**lemma** *mono-on-at-top-le*:

**fixes**  $a :: 'a::\text{linorder}$  **and**  $b :: 'b::\{\text{order-topology, linordered-ab-group-add}\}$

**and**  $f :: 'a \Rightarrow 'b$

**assumes**  $f\text{-mono}: \text{mono-on } \{a..\} f$  **and**  $f\text{-to-l}: (f \longrightarrow l) \text{ at-top}$

**shows**  $\bigwedge x. x \in \{a..\} \implies f x \leq l$

*<proof>*

**corollary** *mono-at-top-le*:

**fixes**  $b :: 'b::\{\text{order-topology, linordered-ab-group-add}\}$  **and**  $f :: 'a::\text{linorder} \Rightarrow 'b$

**assumes**  $\text{mono } f$  **and**  $(f \longrightarrow b) \text{ at-top}$

**shows**  $\bigwedge x. f x \leq b$

*<proof>*

**lemma** *mono-on-at-bot-ge*:

**fixes**  $a :: 'a::\text{linorder}$  **and**  $b :: 'b::\{\text{order-topology, linordered-ab-group-add}\}$

**and**  $f :: 'a \Rightarrow 'b$

**assumes**  $f\text{-mono}: \text{mono-on } \{..a\} f$  **and**  $f\text{-to-l}: (f \longrightarrow l) \text{ at-bot}$

**shows**  $\bigwedge x. x \in \{..a\} \implies f x \geq l$

*<proof>*

**corollary** *mono-at-bot-ge*:

**fixes**  $b :: 'b::\{\text{order-topology, linordered-ab-group-add}\}$  **and**  $f :: 'a::\text{linorder} \Rightarrow 'b$

**assumes**  $\text{mono } f$  **and**  $(f \longrightarrow b) \text{ at-bot}$

**shows**  $\bigwedge x. f x \geq b$

*<proof>*

**lemma** *antimono-on-at-top-ge*:

**fixes**  $a :: 'a::\text{linorder}$  **and**  $b :: 'b::\{\text{order-topology, linordered-ab-group-add}\}$

**and**  $f :: 'a \Rightarrow 'b$

**assumes**  $f\text{-antimono}: \text{antimono-on } \{a..\} f$  **and**  $f\text{-to-l}: (f \longrightarrow l) \text{ at-top}$

**shows**  $\bigwedge x. x \in \{a..\} \implies f x \geq l$

*<proof>*

**corollary** *antimono-at-top-le*:

**fixes**  $b :: 'b::\{\text{order-topology, linordered-ab-group-add}\}$  **and**  $f :: 'a::\text{linorder} \Rightarrow 'b$

**assumes**  $\text{antimono } f$  **and**  $(f \longrightarrow b) \text{ at-top}$

**shows**  $\bigwedge x. f x \geq b$

*<proof>*

**lemma** *antimono-on-at-bot-ge*:

**fixes**  $a :: 'a::\text{linorder}$  **and**  $b :: 'b::\{\text{order-topology, linordered-ab-group-add}\}$

**and**  $f :: 'a \Rightarrow 'b$

**assumes**  $f\text{-antimono}: \text{antimono-on } \{..a\} f$  **and**  $f\text{-to-l}: (f \longrightarrow l) \text{ at-bot}$

**shows**  $\bigwedge x. x \in \{..a\} \implies f x \leq l$

*<proof>*

**corollary** *antimono-at-bot-ge*:

**fixes**  $b :: 'b::\{\text{order-topology, linordered-ab-group-add}\}$  **and**  $f :: 'a::\text{linorder} \Rightarrow 'b$

**assumes** *antimono f* and  $(f \longrightarrow b)$  *at-bot*  
**shows**  $\bigwedge x. f\ x \leq b$   
*<proof>*

**lemma** *continuous-cdivide*:  
**fixes**  $c :: 'a :: \text{real-normed-field}$   
**assumes**  $c \neq 0$  *continuous F f*  
**shows** *continuous F*  $(\lambda x. f\ x / c)$   
*<proof>*

**lemma** *continuous-mult-left-iff*:  
**fixes**  $c :: 'a :: \text{real-normed-field}$   
**assumes**  $c \neq 0$   
**shows** *continuous F f*  $\longleftrightarrow$  *continuous F*  $(\lambda x. c * f\ x)$   
*<proof>*

**lemma** *continuous-mult-right-iff*:  
**fixes**  $c :: 'a :: \text{real-normed-field}$   
**assumes**  $c \neq 0$   
**shows** *continuous F f*  $\longleftrightarrow$  *continuous F*  $(\lambda x. f\ x * c)$   
*<proof>*

**lemma** *continuous-cdivide-iff*:  
**fixes**  $c :: 'a :: \text{real-normed-field}$   
**assumes**  $c \neq 0$   
**shows** *continuous F f*  $\longleftrightarrow$  *continuous F*  $(\lambda x. f\ x / c)$   
*<proof>*

**lemma** *continuous-cong*:  
**assumes** *eventually*  $(\lambda x. f\ x = g\ x)$  *F f*  $(\text{Lim } F\ (\lambda x. x)) = g\ (\text{Lim } F\ (\lambda x. x))$   
**shows** *continuous F f*  $\longleftrightarrow$  *continuous F g*  
*<proof>*

**lemma** *continuous-at-within-cong*:  
**assumes**  $f\ x = g\ x$  *eventually*  $(\lambda x. f\ x = g\ x)$  *(at x within s)*  
**shows** *continuous (at x within s) f*  $\longleftrightarrow$  *continuous (at x within s) g*  
*<proof>*

**lemma** *continuous-within-shift*:  
**fixes**  $a\ x :: 'a :: \{\text{topological-ab-group-add, t2-space}\}$   
**and**  $s :: 'a\ \text{set}$   
**and**  $f :: 'a \Rightarrow 'b :: \text{topological-space}$   
**shows** *continuous (at x within s)*  $(\lambda x. f\ (x+a)) \longleftrightarrow$  *continuous (at (x+a) within*  
*plus a ' s) f*  
*<proof>*

**lemma** *isCont-shift*:  
**fixes**  $a\ x :: 'a :: \{\text{topological-ab-group-add, t2-space}\}$   
**and**  $f :: 'a \Rightarrow 'b :: \text{topological-space}$

**shows**  $isCont (\lambda x. f (x+a)) x \longleftrightarrow isCont f (x+a)$   
 ⟨proof⟩

**lemma** *has-real-derivative-at-split*:

$(f \text{ has-real-derivative } D) (at\ x) \longleftrightarrow$   
 $(f \text{ has-real-derivative } D) (at\text{-left } x) \wedge (f \text{ has-real-derivative } D) (at\text{-right } x)$   
 ⟨proof⟩

**lemma** *DERIV-cmult-iff*:

**assumes**  $c \neq 0$   
**shows**  $(f \text{ has-field-derivative } D) (at\ x \text{ within } s) \longleftrightarrow$   
 $((\lambda x. c * f\ x) \text{ has-field-derivative } c * D) (at\ x \text{ within } s)$   
 ⟨proof⟩

**lemma** *DERIV-cmult-right-iff*:

**assumes**  $c \neq 0$   
**shows**  $(f \text{ has-field-derivative } D) (at\ x \text{ within } s) \longleftrightarrow$   
 $((\lambda x. f\ x * c) \text{ has-field-derivative } D * c) (at\ x \text{ within } s)$   
 ⟨proof⟩

**lemma** *DERIV-cdivide-iff*:

**assumes**  $c \neq 0$   
**shows**  $(f \text{ has-field-derivative } D) (at\ x \text{ within } s) \longleftrightarrow$   
 $((\lambda x. f\ x / c) \text{ has-field-derivative } D / c) (at\ x \text{ within } s)$   
 ⟨proof⟩

**lemma** *DERIV-ln-divide-chain*:

**fixes**  $f :: real \Rightarrow real$   
**assumes**  $f\ x > 0$  **and**  $(f \text{ has-real-derivative } D) (at\ x \text{ within } s)$   
**shows**  $((\lambda x. \ln (f\ x)) \text{ has-real-derivative } (D / f\ x)) (at\ x \text{ within } s)$   
 ⟨proof⟩

**lemma** *inverse-fun-has-integral-ln*:

**fixes**  $f :: real \Rightarrow real$  **and**  $f' :: real \Rightarrow real$   
**assumes**  $a \leq b$  **and**  
 $\bigwedge x. x \in \{a..b\} \implies f\ x > 0$  **and**  
 $continuous\text{-on } \{a..b\} f$  **and**  
 $\bigwedge x. x \in \{a <..<b\} \implies (f \text{ has-real-derivative } f'\ x) (at\ x)$   
**shows**  $((\lambda x. f'\ x / f\ x) \text{ has-integral } (\ln (f\ b) - \ln (f\ a))) \{a..b\}$   
 ⟨proof⟩

**lemma** *DERIV-fun-powr2*:

**fixes**  $a :: real$   
**assumes**  $a\text{-pos}: a > 0$   
**and**  $f: DERIV\ f\ x :=> r$   
**shows**  $DERIV (\lambda x. a. \wedge (f\ x)) x :=> a. \wedge (f\ x) * r * \ln a$   
 ⟨proof⟩

**lemma** *has-real-derivative-powr2*:



**assumes**  $a\text{-pos}$ :  $a > 0$   
**shows**  $((\lambda x. a. \hat{x}) \text{ has-real-derivative } a. \hat{x} * \ln a) (at\ x)$   
 $\langle proof \rangle$

**lemma** *field-differentiable-shift*:  
 $(f \text{ field-differentiable } (at\ (x + z))) = ((\lambda x. f\ (x + z)) \text{ field-differentiable } (at\ x))$   
 $\langle proof \rangle$

## 1.1 Lemmas on *indicator* for a Linearly Ordered Type

**lemma** *indicator-Icc-shift*:  
**fixes**  $a\ b\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{a..b\}\ x = \text{indicator } \{t+a..t+b\}\ (t+x)$   
 $\langle proof \rangle$

**lemma** *indicator-Icc-shift-inverse*:  
**fixes**  $a\ b\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{a-t..b-t\}\ x = \text{indicator } \{a..b\}\ (t+x)$   
 $\langle proof \rangle$

**lemma** *indicator-Ici-shift*:  
**fixes**  $a\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{a.. \}\ x = \text{indicator } \{t+a.. \}\ (t+x)$   
 $\langle proof \rangle$

**lemma** *indicator-Ici-shift-inverse*:  
**fixes**  $a\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{a-t.. \}\ x = \text{indicator } \{a.. \}\ (t+x)$   
 $\langle proof \rangle$

**lemma** *indicator-Iic-shift*:  
**fixes**  $b\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{..b\}\ x = \text{indicator } \{..t+b\}\ (t+x)$   
 $\langle proof \rangle$

**lemma** *indicator-Iic-shift-inverse*:  
**fixes**  $b\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{..b-t\}\ x = \text{indicator } \{..b\}\ (t+x)$   
 $\langle proof \rangle$

**lemma** *indicator-Icc-reverse*:  
**fixes**  $a\ b\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{a..b\}\ x = \text{indicator } \{t-b..t-a\}\ (t-x)$   
 $\langle proof \rangle$

**lemma** *indicator-Icc-reverse-inverse*:  
**fixes**  $a\ b\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{t-b..t-a\}\ x = \text{indicator } \{a..b\}\ (t-x)$

*<proof>*

**lemma** *indicator-Ici-reverse:*

**fixes**  $a\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{a..\} x = \text{indicator } \{..t-a\} (t-x)$   
*<proof>*

**lemma** *indicator-Ici-reverse-inverse:*

**fixes**  $b\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{t-b..\} x = \text{indicator } \{..b\} (t-x)$   
*<proof>*

**lemma** *indicator-Iic-reverse:*

**fixes**  $b\ t\ x :: 'a::\text{linordered-ab-group-add}$   
**shows**  $\text{indicator } \{..b\} x = \text{indicator } \{t-b..\} (t-x)$   
*<proof>*

**lemma** *indicator-Iic-reverse-inverse:*

**fixes**  $a\ t\ x :: 'a::\text{linordered-field}$   
**shows**  $\text{indicator } \{..t-a\} x = \text{indicator } \{a..\} (t-x)$   
*<proof>*

**lemma** *indicator-Icc-affine-pos:*

**fixes**  $a\ b\ c\ t\ x :: 'a::\text{linordered-field}$   
**assumes**  $c > 0$   
**shows**  $\text{indicator } \{a..b\} x = \text{indicator } \{t+c*a..t+c*b\} (t + c*x)$   
*<proof>*

**lemma** *indicator-Icc-affine-pos-inverse:*

**fixes**  $a\ b\ c\ t\ x :: 'a::\text{linordered-field}$   
**assumes**  $c > 0$   
**shows**  $\text{indicator } \{(a-t)/c..(b-t)/c\} x = \text{indicator } \{a..b\} (t + c*x)$   
*<proof>*

**lemma** *indicator-Ici-affine-pos:*

**fixes**  $a\ c\ t\ x :: 'a::\text{linordered-field}$   
**assumes**  $c > 0$   
**shows**  $\text{indicator } \{a..\} x = \text{indicator } \{t+c*a..\} (t + c*x)$   
*<proof>*

**lemma** *indicator-Ici-affine-pos-inverse:*

**fixes**  $a\ c\ t\ x :: 'a::\text{linordered-field}$   
**assumes**  $c > 0$   
**shows**  $\text{indicator } \{(a-t)/c..\} x = \text{indicator } \{a..\} (t + c*x)$   
*<proof>*

**lemma** *indicator-Iic-affine-pos:*

**fixes**  $b\ c\ t\ x :: 'a::\text{linordered-field}$   
**assumes**  $c > 0$

**shows**  $\text{indicator } \{..b\} x = \text{indicator } \{..t+c*b\} (t + c*x)$   
*<proof>*

**lemma** *indicator-Iic-affine-pos-inverse:*

**fixes**  $b\ c\ t\ x :: 'a::\text{linordered-field}$

**assumes**  $c > 0$

**shows**  $\text{indicator } \{..(b-t)/c\} x = \text{indicator } \{..b\} (t + c*x)$

*<proof>*

**lemma** *indicator-Icc-affine-neg:*

**fixes**  $a\ b\ c\ t\ x :: 'a::\text{linordered-field}$

**assumes**  $c < 0$

**shows**  $\text{indicator } \{a..b\} x = \text{indicator } \{t+c*b..t+c*a\} (t + c*x)$

*<proof>*

**lemma** *indicator-Icc-affine-neg-inverse:*

**fixes**  $a\ b\ c\ t\ x :: 'a::\text{linordered-field}$

**assumes**  $c < 0$

**shows**  $\text{indicator } \{(b-t)/c..(a-t)/c\} x = \text{indicator } \{a..b\} (t + c*x)$

*<proof>*

**lemma** *indicator-Ici-affine-neg:*

**fixes**  $a\ c\ t\ x :: 'a::\text{linordered-field}$

**assumes**  $c < 0$

**shows**  $\text{indicator } \{a.. \} x = \text{indicator } \{..t+c*a\} (t + c*x)$

*<proof>*

**lemma** *indicator-Ici-affine-neg-inverse:*

**fixes**  $b\ c\ t\ x :: 'a::\text{linordered-field}$

**assumes**  $c < 0$

**shows**  $\text{indicator } \{(b-t)/c.. \} x = \text{indicator } \{..b\} (t + c*x)$

*<proof>*

**lemma** *indicator-Iic-affine-neg:*

**fixes**  $b\ c\ t\ x :: 'a::\text{linordered-field}$

**assumes**  $c < 0$

**shows**  $\text{indicator } \{..b\} x = \text{indicator } \{t+c*b.. \} (t + c*x)$

*<proof>*

**lemma** *indicator-Iic-affine-neg-inverse:*

**fixes**  $a\ c\ t\ x :: 'a::\text{linordered-field}$

**assumes**  $c < 0$

**shows**  $\text{indicator } \{..(a-t)/c\} x = \text{indicator } \{a.. \} (t + c*x)$

*<proof>*

## 2 Additional Lemmas for the *HOL-Analysis* Library

**lemma** *differentiable-eq-field-differentiable-real:*

**fixes**  $f :: \text{real} \Rightarrow \text{real}$

**shows**  $f$  differentiable  $F \iff f$  field-differentiable  $F$   
(proof)

**lemma** differentiable-on-eq-field-differentiable-real:  
**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
**shows**  $f$  differentiable-on  $s \iff (\forall x \in s. f \text{ field-differentiable (at } x \text{ within } s))$   
(proof)

**lemma** differentiable-on-cong :  
**assumes**  $\bigwedge x. x \in s \implies f x = g x$  **and**  $f$  differentiable-on  $s$   
**shows**  $g$  differentiable-on  $s$   
(proof)

**lemma** C1-differentiable-imp-deriv-continuous-on:  
 $f$  C1-differentiable-on  $S \implies$  continuous-on  $S$  (deriv  $f$ )  
(proof)

**lemma** deriv-shift:  
**assumes**  $f$  field-differentiable at  $(x+a)$   
**shows**  $\text{deriv } f (x+a) = \text{deriv } (\lambda s. f (x+s)) a$   
(proof)

**lemma** piecewise-differentiable-on-cong:  
**assumes**  $f$  piecewise-differentiable-on  $i$   
**and**  $\bigwedge x. x \in i \implies f x = g x$   
**shows**  $g$  piecewise-differentiable-on  $i$   
(proof)

**lemma** differentiable-piecewise:  
**assumes** continuous-on  $i$   $f$   
**and**  $f$  differentiable-on  $i$   
**shows**  $f$  piecewise-differentiable-on  $i$   
(proof)

**lemma** piecewise-differentiable-scaleR:  
**assumes**  $f$  piecewise-differentiable-on  $S$   
**shows**  $(\lambda x. a *_{\mathbb{R}} f x)$  piecewise-differentiable-on  $S$   
(proof)

**lemma** differentiable-on-piecewise-compose:  
**assumes**  $f$  piecewise-differentiable-on  $S$   
**and**  $g$  differentiable-on  $f ' S$   
**shows**  $g \circ f$  piecewise-differentiable-on  $S$   
(proof)

**lemma** MVT-order-free:  
**fixes**  $r h :: \text{real}$   
**defines**  $I \equiv \{r..r+h\} \cup \{r+h..r\}$   
**assumes** continuous-on  $I$   $f$  **and**  $f$  differentiable-on interior  $I$

**obtains  $t$  where  $t \in \{0 < .. < 1\}$  and  $f(r+h) - f r = h * deriv f (r + t*h)$**   
 <proof>

**lemma *integral-combine2*:**

**fixes  $f :: real \Rightarrow 'a::banach$**

**assumes  $a \leq c \ c \leq b$**

**and  $f$  integrable-on  $\{a..c\}$   $f$  integrable-on  $\{c..b\}$**

**shows  $integral \{a..c\} f + integral \{c..b\} f = integral \{a..b\} f$**

<proof>

**lemma *has-integral-null-interval*: fixes  $a \ b :: real$  and  $f::real \Rightarrow real$  assumes  $a \geq b$**

**shows  $(f \text{ has-integral } 0) \{a..b\}$**

<proof>

**lemma *has-integral-interval-reverse*: fixes  $f :: real \Rightarrow real$  and  $a \ b :: real$**

**assumes  $a \leq b$**

**and  $f$  continuous-on  $\{a..b\}$**

**shows  $((\lambda x. f(a+b-x)) \text{ has-integral } (integral \{a..b\} f)) \{a..b\}$**

<proof>

**lemma *FTC-real-deriv-has-integral*:**

**fixes  $F :: real \Rightarrow real$**

**assumes  $a \leq b$**

**and  $F$  piecewise-differentiable-on  $\{a < .. < b\}$**

**and  $f$  continuous-on  $\{a..b\}$**

**shows  $(deriv F \text{ has-integral } F b - F a) \{a..b\}$**

<proof>

**lemma *integrable-spike-cong*:**

**assumes negligible  $S \ \wedge x. x \in T - S \implies g x = f x$**

**shows  $f$  integrable-on  $T \iff g$  integrable-on  $T$**

<proof>

**lemma *has-integral-powr2-from-0*:**

**fixes  $a \ c :: real$**

**assumes  $a$ -pos:  $a > 0$  and  $a$ -neg-1:  $a \neq 1$  and  $c$ -nneg:  $c \geq 0$**

**shows  $((\lambda x. a.^{\hat{x}}) \text{ has-integral } ((a.^{\hat{c}} - 1) / (\ln a))) \{0..c\}$**

<proof>

**lemma *integrable-on-powr2-from-0*:**

**fixes  $a \ c :: real$**

**assumes  $a$ -pos:  $a > 0$  and  $a$ -neg-1:  $a \neq 1$  and  $c$ -nneg:  $c \geq 0$**

**shows  $(\lambda x. a.^{\hat{x}}) \text{ integrable-on } \{0..c\}$**

<proof>

**lemma *integrable-on-powr2-from-0-general*:**

**fixes  $a \ c :: real$**

**assumes  $a$ -pos:  $a > 0$  and  $c$ -nneg:  $c \geq 0$**

**shows**  $(\lambda x. a. \hat{x})$  integrable-on  $\{0..c\}$   
*<proof>*

**lemma** *has-bochner-integral-power*:

**fixes**  $a b :: \text{real}$  **and**  $k :: \text{nat}$

**assumes**  $a \leq b$

**shows** *has-bochner-integral lborel*  $(\lambda x. x^k * \text{indicator } \{a..b\} x)$   $((b^{k+1} - a^{k+1}) / (k+1))$

*<proof>*

**corollary** *integrable-power*:  $(a :: \text{real}) \leq b \implies \text{integrable lborel } (\lambda x. x^k * \text{indicator } \{a..b\} x)$

*<proof>*

**lemma** *has-integral-set-integral-real*:

**fixes**  $f :: 'a :: \text{euclidean-space} \Rightarrow \text{real}$  **and**  $A :: 'a \text{ set}$

**assumes**  $f$ : *set-integrable lborel*  $A f$

**shows**  $(f \text{ has-integral } (\text{set-lebesgue-integral lborel } A f)) A$

*<proof>*

**lemma** *set-borel-measurable-lborel*:

*set-borel-measurable lborel*  $A f \longleftrightarrow \text{set-borel-measurable borel } A f$

*<proof>*

**lemma** *restrict-space-whole[simp]*: *restrict-space*  $M$  (*space*  $M$ ) =  $M$

*<proof>*

**lemma** *deriv-measurable-real*:

**fixes**  $f :: \text{real} \Rightarrow \text{real}$

**assumes**  $f$  *differentiable-on*  $S$  *open*  $S f \in \text{borel-measurable borel}$

**shows** *set-borel-measurable borel*  $S$  (*deriv*  $f$ )

*<proof>*

**lemma** *piecewise-differentiable-on-deriv-measurable-real*:

**fixes**  $f :: \text{real} \Rightarrow \text{real}$

**assumes**  $f$  *piecewise-differentiable-on*  $S$  *open*  $S f \in \text{borel-measurable borel}$

**shows** *set-borel-measurable borel*  $S$  (*deriv*  $f$ )

*<proof>*

**lemma** *borel-measurable-antimono*:

**fixes**  $f :: \text{real} \Rightarrow \text{real}$

**shows** *antimono*  $f \implies f \in \text{borel-measurable borel}$

*<proof>*

**lemma** *set-borel-measurable-restrict-space-iff*:

**fixes**  $f :: 'a \Rightarrow 'b :: \text{real-normed-vector}$

**assumes**  $\Omega[\text{measurable, simp}]$ :  $\Omega \cap \text{space } M \in \text{sets } M$

**shows**  $f \in \text{borel-measurable } (\text{restrict-space } M \ \Omega) \longleftrightarrow \text{set-borel-measurable } M \ \Omega$   
 $f$   
 $\langle \text{proof} \rangle$

**lemma** *set-integrable-restrict-space-iff*:  
**fixes**  $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$   
**assumes**  $A \in \text{sets } M$   
**shows**  $\text{set-integrable } M \ A \ f \longleftrightarrow \text{integrable } (\text{restrict-space } M \ A) \ f$   
 $\langle \text{proof} \rangle$

**lemma** *set-lebesgue-integral-restrict-space*:  
**fixes**  $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$   
**assumes**  $A \in \text{sets } M$   
**shows**  $\text{set-lebesgue-integral } M \ A \ f = \text{integral}^L (\text{restrict-space } M \ A) \ f$   
 $\langle \text{proof} \rangle$

**lemma** *distr-borel-lborel*:  $\text{distr } M \ \text{borel } f = \text{distr } M \ \text{lborel } f$   
 $\langle \text{proof} \rangle$

**lemma** *AE-translation*:  
**assumes**  $\text{AE } x \text{ in } \text{lborel}. P \ x$  **shows**  $\text{AE } x \text{ in } \text{lborel}. P \ (a+x)$   
 $\langle \text{proof} \rangle$

**lemma** *set-AE-translation*:  
**assumes**  $\text{AE } x \in S \text{ in } \text{lborel}. P \ x$  **shows**  $\text{AE } x \in \text{plus } (-a) \ 'S \text{ in } \text{lborel}. P \ (a+x)$   
 $\langle \text{proof} \rangle$

**lemma** *AE-scale-measure-iff*:  
**assumes**  $r > 0$   
**shows**  $(\text{AE } x \text{ in } (\text{scale-measure } r \ M). P \ x) \longleftrightarrow (\text{AE } x \text{ in } M. P \ x)$   
 $\langle \text{proof} \rangle$

**lemma** *nn-set-integral-cong2*:  
**assumes**  $\text{AE } x \in A \text{ in } M. f \ x = g \ x$   
**shows**  $(\int^{+x \in A}. f \ x \ \partial M) = (\int^{+x \in A}. g \ x \ \partial M)$   
 $\langle \text{proof} \rangle$

**lemma** *set-lebesgue-integral-cong-AE2*:  
**assumes**  $[\text{measurable}]: A \in \text{sets } M \ \text{set-borel-measurable } M \ A \ f \ \text{set-borel-measurable } M \ A \ g$   
**assumes**  $\text{AE } x \in A \text{ in } M. f \ x = g \ x$   
**shows**  $(\text{LINT } x:A|M. f \ x) = (\text{LINT } x:A|M. g \ x)$   
 $\langle \text{proof} \rangle$

**proposition** *set-nn-integral-eq-set-integral*:  
**assumes**  $\text{AE } x \in A \text{ in } M. 0 \leq f \ x \ \text{set-integrable } M \ A \ f$   
**shows**  $(\int^{+x \in A}. f \ x \ \partial M) = (\int x \in A. f \ x \ \partial M)$   
 $\langle \text{proof} \rangle$

**proposition** *nn-integral-disjoint-family-on-finite:*

**assumes** [*measurable*]:  $f \in \text{borel-measurable } M \wedge (n::\text{nat}). n \in S \implies B n \in \text{sets } M$

**and** *disjoint-family-on*  $B S$  *finite*  $S$

**shows**  $(\int^+ x \in (\bigcup_{n \in S} B n). f x \partial M) = (\sum_{n \in S}. (\int^+ x \in B n. f x \partial M))$   
*<proof>*

**lemma** *nn-integral-distr-set:*

**assumes**  $T \in \text{measurable } M M'$  **and**  $f \in \text{borel-measurable } (\text{distr } M M' T)$

**and**  $A \in \text{sets } M'$  **and**  $\bigwedge x. x \in \text{space } M \implies T x \in A$

**shows**  $\text{integral}^N (\text{distr } M M' T) f = \text{set-nn-integral } (\text{distr } M M' T) A f$   
*<proof>*

**lemma** *measure-eqI-Ioc:*

**fixes**  $M N :: \text{real measure}$

**assumes** *sets*:  $\text{sets } M = \text{sets borel sets } N = \text{borel}$

**assumes** *fin*:  $\bigwedge a b. a \leq b \implies \text{emeasure } M \{a <.. b\} < \infty$

**assumes** *eq*:  $\bigwedge a b. a \leq b \implies \text{emeasure } M \{a <.. b\} = \text{emeasure } N \{a <.. b\}$

**shows**  $M = N$

*<proof>*

**lemma** (*in finite-measure*) *distributed-measure:*

**assumes** *distributed*  $M N X f$

**and**  $\bigwedge x. x \in \text{space } N \implies f x \geq 0$

**and**  $A \in \text{sets } N$

**shows**  $\text{measure } M (X -' A \cap \text{space } M) = (\int x. \text{indicator } A x * f x \partial N)$

*<proof>*

**lemma** *set-integrable-const[simp]:*

$A \in \text{sets } M \implies \text{emeasure } M A < \infty \implies \text{set-integrable } M A (\lambda-. c)$

*<proof>*

**lemma** *set-integral-const[simp]:*

$A \in \text{sets } M \implies \text{emeasure } M A < \infty \implies \text{set-lebesgue-integral } M A (\lambda-. c) = \text{measure } M A *_R c$

*<proof>*

**lemma** *set-integral-empty-0[simp]:*  $\text{set-lebesgue-integral } M \{\} f = 0$

*<proof>*

**lemma** *set-integral-nonneg[simp]:*

**fixes**  $f :: 'a \Rightarrow \text{real}$  **and**  $A :: 'a \text{ set}$

**shows**  $(\bigwedge x. x \in A \implies 0 \leq f x) \implies 0 \leq \text{set-lebesgue-integral } M A f$

*<proof>*

**lemma**

**fixes**  $f :: 'a \Rightarrow 'b::\{\text{banach, second-countable-topology}\}$  **and**  $w :: 'a \Rightarrow \text{real}$



**assumes**  $A \in \text{sets } M \text{ set-borel-measurable } M A f$   
 $\bigwedge i. \text{ set-borel-measurable } M A (s i) \text{ set-integrable } M A w$   
**assumes**  $\text{lim}: AE x \in A \text{ in } M. (\lambda i. s i x) \longrightarrow f x$   
**assumes**  $\text{bound}: \bigwedge i::\text{nat}. AE x \in A \text{ in } M. \text{norm } (s i x) \leq w x$   
**shows**  $\text{set-integrable-dominated-convergence}: \text{set-integrable } M A f$   
**and**  $\text{set-integrable-dominated-convergence2}: \bigwedge i. \text{set-integrable } M A (s i)$   
**and**  $\text{set-integral-dominated-convergence}: (\lambda i. \text{set-lebesgue-integral } M A (s i)) \longrightarrow \text{set-lebesgue-integral } M A f$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{absolutely-integrable-on-iff-set-integrable}: \text{fixes } f :: 'a::\text{euclidean-space} \Rightarrow \text{real}$   
**assumes**  $f \in \text{borel-measurable lborel}$   
**and**  $S \in \text{sets lborel}$   
**shows**  $\text{set-integrable lborel } S f \longleftrightarrow f \text{ absolutely-integrable-on } S$   
 $\langle \text{proof} \rangle$

**corollary**  $\text{integrable-on-iff-set-integrable-nonneg}: \text{fixes } f :: 'a::\text{euclidean-space} \Rightarrow \text{real}$   
**assumes**  $\bigwedge x. x \in S \implies f x \geq 0 f \in \text{borel-measurable lborel}$   
**and**  $S \in \text{sets lborel}$   
**shows**  $\text{set-integrable lborel } S f \longleftrightarrow f \text{ integrable-on } S$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{integrable-on-iff-set-integrable-nonneg-AE}: \text{fixes } f :: 'a::\text{euclidean-space} \Rightarrow \text{real}$   
**assumes**  $AE x \in S \text{ in lborel}. f x \geq 0 f \in \text{borel-measurable lborel}$   
**and**  $S \in \text{sets lborel}$   
**shows**  $\text{set-integrable lborel } S f \longleftrightarrow f \text{ integrable-on } S$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{set-borel-integral-eq-integral-nonneg}: \text{fixes } f :: 'a::\text{euclidean-space} \Rightarrow \text{real}$   
**assumes**  $\bigwedge x. x \in S \implies f x \geq 0 f \in \text{borel-measurable borel } S \in \text{sets borel}$   
**shows**  $(\text{LINT } x : S \mid \text{lborel}. f x) = \text{integral } S f$   
 — Note that  $0 = 0$  holds when the integral diverges.  
 $\langle \text{proof} \rangle$

**lemma**  $\text{set-borel-integral-eq-integral-nonneg-AE}: \text{fixes } f :: 'a::\text{euclidean-space} \Rightarrow \text{real}$   
**assumes**  $AE x \in S \text{ in lborel}. f x \geq 0 f \in \text{borel-measurable borel } S \in \text{sets borel}$   
**shows**  $(\text{LINT } x : S \mid \text{lborel}. f x) = \text{integral } S f$   
 — Note that  $0 = 0$  holds when the integral diverges.  
 $\langle \text{proof} \rangle$

## 2.1 Set Lebesgue Integrability on Affine Transformation

**lemma**  $\text{set-integrable-Icc-affine-pos-iff}: \text{fixes } f :: \text{real} \Rightarrow 'a::\{\text{banach, second-countable-topology}\} \text{ and } a b c t :: \text{real}$

**assumes**  $c > 0$   
**shows** *set-integrable lborel*  $\{(a-t)/c..(b-t)/c\}$   $(\lambda x. f (t + c*x))$   
 $\longleftrightarrow$  *set-integrable lborel*  $\{a..b\}$   $f$   
 $\langle$ *proof* $\rangle$

**corollary** *set-integrable-Icc-shift:*

**fixes**  $f :: \text{real} \Rightarrow 'a::\{\text{banach, second-countable-topology}\}$  **and**  $a\ b\ t :: \text{real}$   
**shows** *set-integrable lborel*  $\{a-t..b-t\}$   $(\lambda x. f (t+x)) \longleftrightarrow$  *set-integrable lborel*  
 $\{a..b\}$   $f$   
 $\langle$ *proof* $\rangle$

**lemma** *set-integrable-Ici-affine-pos-iff:*

**fixes**  $f :: \text{real} \Rightarrow 'a::\{\text{banach, second-countable-topology}\}$  **and**  $a\ c\ t :: \text{real}$   
**assumes**  $c > 0$   
**shows** *set-integrable lborel*  $\{(a-t)/c.. \}$   $(\lambda x. f (t + c*x))$   
 $\longleftrightarrow$  *set-integrable lborel*  $\{a.. \}$   $f$   
 $\langle$ *proof* $\rangle$

**corollary** *set-integrable-Ici-shift:*

**fixes**  $f :: \text{real} \Rightarrow 'a::\{\text{banach, second-countable-topology}\}$  **and**  $a\ t :: \text{real}$   
**shows** *set-integrable lborel*  $\{a-t.. \}$   $(\lambda x. f (t+x)) \longleftrightarrow$  *set-integrable lborel*  $\{a.. \}$   $f$   
 $\langle$ *proof* $\rangle$

**lemma** *set-integrable-Iic-affine-pos-iff:*

**fixes**  $f :: \text{real} \Rightarrow 'a::\{\text{banach, second-countable-topology}\}$  **and**  $b\ c\ t :: \text{real}$   
**assumes**  $c > 0$   
**shows** *set-integrable lborel*  $\{..(b-t)/c\}$   $(\lambda x. f (t + c*x))$   
 $\longleftrightarrow$  *set-integrable lborel*  $\{..b\}$   $f$   
 $\langle$ *proof* $\rangle$

**corollary** *set-integrable-Iic-shift:*

**fixes**  $f :: \text{real} \Rightarrow 'a::\{\text{banach, second-countable-topology}\}$  **and**  $b\ t :: \text{real}$   
**shows** *set-integrable lborel*  $\{..b-t\}$   $(\lambda x. f (t+x)) \longleftrightarrow$  *set-integrable lborel*  $\{..b\}$   $f$   
 $\langle$ *proof* $\rangle$

**lemma** *set-integrable-Icc-affine-neg-iff:*

**fixes**  $f :: \text{real} \Rightarrow 'a::\{\text{banach, second-countable-topology}\}$  **and**  $a\ b\ c\ t :: \text{real}$   
**assumes**  $c < 0$   
**shows** *set-integrable lborel*  $\{(b-t)/c..(a-t)/c\}$   $(\lambda x. f (t + c*x))$   
 $\longleftrightarrow$  *set-integrable lborel*  $\{a..b\}$   $f$   
 $\langle$ *proof* $\rangle$

**corollary** *set-integrable-Icc-reverse:*

**fixes**  $f :: \text{real} \Rightarrow 'a::\{\text{banach, second-countable-topology}\}$  **and**  $a\ b\ t :: \text{real}$   
**shows** *set-integrable lborel*  $\{t-b..t-a\}$   $(\lambda x. f (t-x)) \longleftrightarrow$  *set-integrable lborel*  
 $\{a..b\}$   $f$   
 $\langle$ *proof* $\rangle$

**lemma** *set-integrable-Ici-affine-neg-iff:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $b\ c\ t :: \text{real}$   
**assumes**  $c < 0$   
**shows**  $\text{set-integrable lborel } \{(b-t)/c..\}$   $(\lambda x. f (t + c*x))$   
 $\longleftrightarrow \text{set-integrable lborel } \{..b\}$   $f$   
 $\langle \text{proof} \rangle$

**corollary** *set-integrable-Ici-reverse:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $b\ t :: \text{real}$   
**shows**  $\text{set-integrable lborel } \{t-b..\}$   $(\lambda x. f (t-x)) \longleftrightarrow \text{set-integrable lborel } \{..b\}$   $f$   
 $\langle \text{proof} \rangle$

**lemma** *set-integrable-Iic-affine-neg-iff:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ c\ t :: \text{real}$   
**assumes**  $c < 0$   
**shows**  $\text{set-integrable lborel } \{..(a-t)/c\}$   $(\lambda x. f (t + c*x))$   
 $\longleftrightarrow \text{set-integrable lborel } \{a..\}$   $f$   
 $\langle \text{proof} \rangle$

**corollary** *set-integrable-Iic-reverse:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ t :: \text{real}$   
**shows**  $\text{set-integrable lborel } \{..t-a\}$   $(\lambda x. f (t-x)) \longleftrightarrow \text{set-integrable lborel } \{a..\}$   $f$   
 $\langle \text{proof} \rangle$

## 2.2 Set Lebesgue Integral on Affine Transformation

**lemma** *lborel-set-integral-Icc-affine-pos:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ b\ c :: \text{real}$   
**assumes**  $c > 0$   
**shows**  $(\int x \in \{a..b\}. f\ x\ \partial \text{lborel}) = c *_{\mathbb{R}} (\int x \in \{(a-t)/c..(b-t)/c\}. f (t + c*x)$   
 $\partial \text{lborel})$   
 $\langle \text{proof} \rangle$

**corollary** *lborel-set-integral-Icc-shift:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ b :: \text{real}$   
**shows**  $(\int x \in \{a..b\}. f\ x\ \partial \text{lborel}) = (\int x \in \{a-t..b-t\}. f (t+x)\ \partial \text{lborel})$   
 $\langle \text{proof} \rangle$

**lemma** *lborel-set-integral-Ici-affine-pos:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ c :: \text{real}$   
**assumes**  $c > 0$   
**shows**  $(\int x \in \{a..\}. f\ x\ \partial \text{lborel}) = c *_{\mathbb{R}} (\int x \in \{(a-t)/c..\}. f (t + c*x)\ \partial \text{lborel})$   
 $\langle \text{proof} \rangle$

**corollary** *lborel-set-integral-Ici-shift:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a :: \text{real}$   
**shows**  $(\int x \in \{a..\}. f\ x\ \partial \text{lborel}) = (\int x \in \{a-t..\}. f (t+x)\ \partial \text{lborel})$   
 $\langle \text{proof} \rangle$

**lemma** *lborel-set-integral-Iic-affine-pos:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $b\ c :: \text{real}$   
**assumes**  $c > 0$   
**shows**  $(\int x \in \{..b\}. f\ x\ \partial\text{lborel}) = c *_R (\int x \in \{..(b-t)/c\}. f\ (t + c*x)\ \partial\text{lborel})$   
 $\langle \text{proof} \rangle$

**corollary** *lborel-set-integral-Iic-shift:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $b :: \text{real}$   
**shows**  $(\int x \in \{..b\}. f\ x\ \partial\text{lborel}) = (\int x \in \{..b-t\}. f\ (t+x)\ \partial\text{lborel})$   
 $\langle \text{proof} \rangle$

**lemma** *lborel-set-integral-Icc-affine-neg:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ b\ c :: \text{real}$   
**assumes**  $c < 0$   
**shows**  $(\int x \in \{a..b\}. f\ x\ \partial\text{lborel}) = -c *_R (\int x \in \{(b-t)/c..(a-t)/c\}. f\ (t + c*x)\ \partial\text{lborel})$   
 $\langle \text{proof} \rangle$

**corollary** *lborel-set-integral-Icc-reverse:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ b :: \text{real}$   
**shows**  $(\int x \in \{a..b\}. f\ x\ \partial\text{lborel}) = (\int x \in \{t-b..t-a\}. f\ (t-x)\ \partial\text{lborel})$   
 $\langle \text{proof} \rangle$

**lemma** *lborel-set-integral-Ici-affine-neg:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $b\ c :: \text{real}$   
**assumes**  $c < 0$   
**shows**  $(\int x \in \{..b\}. f\ x\ \partial\text{lborel}) = -c *_R (\int x \in \{(b-t)/c.. \}. f\ (t + c*x)\ \partial\text{lborel})$   
 $\langle \text{proof} \rangle$

**corollary** *lborel-set-integral-Ici-reverse:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $b :: \text{real}$   
**shows**  $(\int x \in \{..b\}. f\ x\ \partial\text{lborel}) = (\int x \in \{t-b.. \}. f\ (t-x)\ \partial\text{lborel})$   
 $\langle \text{proof} \rangle$

**lemma** *lborel-set-integral-Iic-affine-neg:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ c :: \text{real}$   
**assumes**  $c < 0$   
**shows**  $(\int x \in \{a.. \}. f\ x\ \partial\text{lborel}) = -c *_R (\int x \in \{..(a-t)/c\}. f\ (t + c*x)\ \partial\text{lborel})$   
 $\langle \text{proof} \rangle$

**corollary** *lborel-set-integral-Iic-reverse:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a :: \text{real}$   
**shows**  $(\int x \in \{a.. \}. f\ x\ \partial\text{lborel}) = (\int x \in \{..t-a\}. f\ (t-x)\ \partial\text{lborel})$   
 $\langle \text{proof} \rangle$

**lemma** *set-integrable-Ici-equiv-aux:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ b :: \text{real}$   
**assumes**  $\bigwedge c\ d. \text{set-integrable lborel } \{c..d\} \ f\ a \leq b$   
**shows**  $\text{set-integrable lborel } \{a.. \} \ f \longleftrightarrow \text{set-integrable lborel } \{b.. \} \ f$   
 $\langle \text{proof} \rangle$

**corollary** *set-integrable-Ici-equiv:*

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{banach, second-countable-topology}\}$  **and**  $a\ b :: \text{real}$   
**assumes**  $\bigwedge c\ d. \text{set-integrable lborel } \{c..d\} f$   
**shows**  $\text{set-integrable lborel } \{a.. \} f \longleftrightarrow \text{set-integrable lborel } \{b.. \} f$   
 $\langle \text{proof} \rangle$

**lemma** *set-integrable-Iic-equiv:*

**fixes**  $f :: \text{real} \Rightarrow \text{real}$  **and**  $a\ b :: \text{real}$   
**assumes**  $\bigwedge c\ d. \text{set-integrable lborel } \{c..d\} f$   
**shows**  $\text{set-integrable lborel } \{..a\} f \longleftrightarrow \text{set-integrable lborel } \{..b\} f$  (**is** ?LHS  $\longleftrightarrow$  ?RHS)  
 $\langle \text{proof} \rangle$

## 2.3 Alternative Integral Test

**lemma** *nn-integral-suminf-Ico-real-nat:*

**fixes**  $a :: \text{real}$  **and**  $f :: \text{real} \Rightarrow \text{ennreal}$   
**assumes**  $f \in \text{borel-measurable lborel}$   
**shows**  $(\int^{+x \in \{a.. \}} f\ x\ \partial \text{lborel}) = (\sum k. \int^{+x \in \{a+k.. <a+k+1\}} f\ x\ \partial \text{lborel})$   
 $\langle \text{proof} \rangle$

**lemma** *set-integrable-iff-bounded:*

**fixes**  $f :: 'a \Rightarrow 'b :: \{\text{banach, second-countable-topology}\}$   
**assumes**  $A \in \text{sets } M$   
**shows**  $\text{set-integrable } M\ A\ f \longleftrightarrow \text{set-borel-measurable } M\ A\ f \wedge (\int^{+x \in A} \text{norm } (f\ x)\ \partial M) < \infty$   
 $\langle \text{proof} \rangle$

**theorem** *set-integrable-iff-summable:*

**fixes**  $a :: \text{real}$  **and**  $f :: \text{real} \Rightarrow \text{real}$   
**assumes** *antimono-on*  $\{a.. \} f \wedge x. a \leq x \implies f\ x \geq 0$   $f \in \text{borel-measurable lborel}$   
**shows**  $\text{set-integrable lborel } \{a.. \} f \longleftrightarrow \text{summable } (\lambda k. f\ (a+k))$   
 $\langle \text{proof} \rangle$

## 2.4 Interchange of Differentiation and Lebesgue Integration

**definition** *measurable-extension*  $:: 'a\ \text{measure} \Rightarrow 'b\ \text{measure} \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$  **where**

$\text{measurable-extension } M\ N\ f =$   
 $(\text{SOME } g. g \in M \rightarrow_M N \wedge (\exists S \in (\text{null-sets } M). \{x \in \text{space } M. f\ x \neq g\ x\} \subseteq S))$

- The term *measurable-extension* is proposed by Reynald Affeldt.
- This function is used to make an almost-everywhere-defined function measurable.

**lemma**

**fixes**  $f\ g$   
**assumes**  $g \in M \rightarrow_M N$   $S \in \text{null-sets } M$   $\{x \in \text{space } M. f\ x \neq g\ x\} \subseteq S$

**shows** *measurable-extensionI*:  $\text{AE } x \text{ in } M. f x = \text{measurable-extension } M N f x$   
**and**  
*measurable-extensionI2*:  $\text{AE } x \text{ in } M. g x = \text{measurable-extension } M N f x$  **and**  
*measurable-extension-measurable*:  $\text{measurable-extension } M N f \in \text{measurable } M N$   
*<proof>*

**corollary** *measurable-measurable-extension-AE*:  
**fixes**  $f$   
**assumes**  $f \in M \rightarrow_M N$   
**shows**  $\text{AE } x \text{ in } M. f x = \text{measurable-extension } M N f x$   
*<proof>*

**definition** *borel-measurable-extension* ::  
 $'a \text{ measure} \Rightarrow ('a \Rightarrow 'b::\text{topological-space}) \Rightarrow 'a \Rightarrow 'b$  **where**  
*borel-measurable-extension*  $M f = \text{measurable-extension } M \text{ borel } f$

**lemma**  
**fixes**  $f g$   
**assumes**  $g \in \text{borel-measurable } M S \in \text{null-sets } M \{x \in \text{space } M. f x \neq g x\} \subseteq S$   
**shows** *borel-measurable-extensionI*:  $\text{AE } x \text{ in } M. f x = \text{borel-measurable-extension } M f x$  **and**  
*borel-measurable-extensionI2*:  $\text{AE } x \text{ in } M. g x = \text{borel-measurable-extension } M f x$  **and**  
*borel-measurable-extension-measurable*:  $\text{borel-measurable-extension } M f \in \text{borel-measurable } M$   
*<proof>*

**corollary** *borel-measurable-measurable-extension-AE*:  
**fixes**  $f$   
**assumes**  $f \in \text{borel-measurable } M$   
**shows**  $\text{AE } x \text{ in } M. f x = \text{borel-measurable-extension } M f x$   
*<proof>*

**definition** *set-borel-measurable-extension* ::  
 $'a \text{ measure} \Rightarrow 'a \text{ set} \Rightarrow ('a \Rightarrow 'b::\text{topological-space}) \Rightarrow 'a \Rightarrow 'b$   
**where** *set-borel-measurable-extension*  $M A f = \text{borel-measurable-extension } (\text{restrict-space } M A) f$

**lemma**  
**fixes**  $f g :: 'a \Rightarrow 'b::\text{real-normed-vector}$  **and**  $A$   
**assumes**  $A \in \text{sets } M \text{ set-borel-measurable } M A g S \in \text{null-sets } M \{x \in A. f x \neq g x\} \subseteq S$   
**shows** *set-borel-measurable-extensionI*:  
 $\text{AE } x \in A \text{ in } M. f x = \text{set-borel-measurable-extension } M A f x$  **and**  
*set-borel-measurable-extensionI2*:  
 $\text{AE } x \in A \text{ in } M. g x = \text{set-borel-measurable-extension } M A f x$  **and**  
*set-borel-measurable-extension-measurable*:  
 $\text{set-borel-measurable } M A (\text{set-borel-measurable-extension } M A f)$

*<proof>*

**corollary** *set-borel-measurable-measurable-extension-AE:*

**fixes**  $f :: 'a \Rightarrow 'b :: \text{real-normed-vector}$  **and**  $A$

**assumes** *set-borel-measurable*  $M A f A \in \text{sets } M$

**shows**  $AE x \in A$  in  $M$ .  $f x = \text{set-borel-measurable-extension } M A f x$

*<proof>*

**proposition** *interchange-deriv-LINT-general:*

**fixes**  $a b :: \text{real}$  **and**  $f :: \text{real} \Rightarrow 'a \Rightarrow \text{real}$  **and**  $g :: 'a \Rightarrow \text{real}$

**assumes** *f-integ*:  $\bigwedge r. r \in \{a <..< b\} \implies \text{integrable } M (f r)$  **and**

*f-diff*:  $AE x$  in  $M$ .  $(\lambda r. f r x)$  *differentiable-on*  $\{a <..< b\}$  **and**

*Df-bound*:  $AE x$  in  $M$ .  $\forall r \in \{a <..< b\}. |\text{deriv } (\lambda r. f r x) r| \leq g x$  *integrable*  $M g$

**shows**  $\bigwedge r. r \in \{a <..< b\} \implies ((\lambda r. \int x. f r x \partial M)$  *has-real-derivative*

$\int x. \text{borel-measurable-extension } M (\lambda x. \text{deriv } (\lambda r. f r x) r) x \partial M)$  (at  $r$ )

*<proof>*

**proposition** *interchange-deriv-LINT:*

**fixes**  $a b :: \text{real}$  **and**  $f :: \text{real} \Rightarrow 'a \Rightarrow \text{real}$  **and**  $g :: 'a \Rightarrow \text{real}$

**assumes**  $\bigwedge r. r \in \{a <..< b\} \implies \text{integrable } M (f r)$  **and**

$AE x$  in  $M$ .  $(\lambda r. f r x)$  *differentiable-on*  $\{a <..< b\}$  **and**

$\bigwedge r. r \in \{a <..< b\} \implies (\lambda x. (\text{deriv } (\lambda r. f r x) r)) \in \text{borel-measurable } M$  **and**

$AE x$  in  $M$ .  $\forall r \in \{a <..< b\}. |\text{deriv } (\lambda r. f r x) r| \leq g x$  *integrable*  $M g$

**shows**  $\bigwedge r. r \in \{a <..< b\} \implies ((\lambda r. \int x. f r x \partial M)$  *has-real-derivative*

$\int x. \text{deriv } (\lambda r. f r x) r \partial M)$  (at  $r$ )

*<proof>*

**proposition** *interchange-deriv-LINT-set-general:*

**fixes**  $a b :: \text{real}$  **and**  $f :: \text{real} \Rightarrow 'a \Rightarrow \text{real}$  **and**  $g :: 'a \Rightarrow \text{real}$  **and**  $A :: 'a \text{ set}$

**assumes** *A-msr*:  $A \in \text{sets } M$  **and**

*f-integ*:  $\bigwedge r. r \in \{a <..< b\} \implies \text{set-integrable } M A (f r)$  **and**

*f-diff*:  $AE x \in A$  in  $M$ .  $(\lambda r. f r x)$  *differentiable-on*  $\{a <..< b\}$  **and**

*Df-bound*:  $AE x \in A$  in  $M$ .  $\forall r \in \{a <..< b\}. |\text{deriv } (\lambda r. f r x) r| \leq g x$  *set-integrable*

$M A g$

**shows**  $\bigwedge r. r \in \{a <..< b\} \implies ((\lambda r. \int x \in A. f r x \partial M)$  *has-real-derivative*

$(\int x \in A. \text{set-borel-measurable-extension } M A (\lambda x. \text{deriv } (\lambda r. f r x) r) x \partial M))$

(at  $r$ )

*<proof>*

**proposition** *interchange-deriv-LINT-set:*

**fixes**  $a b :: \text{real}$  **and**  $f :: \text{real} \Rightarrow 'a \Rightarrow \text{real}$  **and**  $g :: 'a \Rightarrow \text{real}$  **and**  $A :: 'a \text{ set}$

**assumes**  $A \in \text{sets } M$  **and**

$\bigwedge r. r \in \{a <..< b\} \implies \text{set-integrable } M A (f r)$  **and**

$AE x \in A$  in  $M$ .  $(\lambda r. f r x)$  *differentiable-on*  $\{a <..< b\}$  **and**

$\bigwedge r. r \in \{a <..< b\} \implies \text{set-borel-measurable } M A (\lambda x. (\text{deriv } (\lambda r. f r x) r))$  **and**

$AE x \in A$  in  $M$ .  $\forall r \in \{a <..< b\}. |\text{deriv } (\lambda r. f r x) r| \leq g x$  *set-integrable*  $M A g$

**shows**  $\bigwedge r. r \in \{a <..< b\} \implies ((\lambda r. \int x \in A. f r x \partial M)$  *has-real-derivative*

$(\int x \in A. \text{deriv } (\lambda r. f r x) r \partial M))$  (at  $r$ )

*<proof>*

### 3 Additional Lemmas for the *HOL-Probability* Library

**lemma** (in *finite-borel-measure*)

**fixes**  $F :: \text{real} \Rightarrow \text{real}$

**assumes**  $\text{nondec} F : \bigwedge x y. x \leq y \implies F x \leq F y$  **and**

$\text{right-cont-}F : \bigwedge a. \text{continuous (at-right } a) F$  **and**

$\text{lim-}F\text{-at-bot} : (F \longrightarrow 0) \text{ at-bot}$  **and**

$\text{lim-}F\text{-at-top} : (F \longrightarrow m) \text{ at-top}$  **and**

$m : 0 \leq m$

**shows**  $\text{emeasure-interval-measure-Ioi: emeasure (interval-measure } F) \{x < ..\} = m - F x$

**and**  $\text{measure-interval-measure-Ioi: measure (interval-measure } F) \{x < ..\} = m - F x$

*<proof>*

**lemma** (in *prob-space*)  $\text{cond-prob-nonneg[simp]: cond-prob } M P Q \geq 0$

*<proof>*

**lemma** (in *prob-space*)  $\text{cond-prob-whole-1: cond-prob } M P P = 1$  **if**  $\text{prob } \{\omega \in \text{space } M. P \omega\} \neq 0$

*<proof>*

**lemma** (in *prob-space*)  $\text{cond-prob-0-null: cond-prob } M P Q = 0$  **if**  $\text{prob } \{\omega \in \text{space } M. Q \omega\} = 0$

*<proof>*

**lemma** (in *prob-space*)  $\text{cond-prob-AE-prob:}$

**assumes**  $\{\omega \in \text{space } M. P \omega\} \in \text{events}$   $\{\omega \in \text{space } M. Q \omega\} \in \text{events}$

**and**  $\text{AE } \omega \text{ in } M. Q \omega$

**shows**  $\text{cond-prob } M P Q = \text{prob } \{\omega \in \text{space } M. P \omega\}$

*<proof>*

#### 3.1 More Properties of *cdf*'s

**context** *finite-borel-measure*

**begin**

**lemma**  $\text{cdf-diff-eq2:}$

**assumes**  $x \leq y$

**shows**  $\text{cdf } M y - \text{cdf } M x = \text{measure } M \{x < ..y\}$

*<proof>*

**end**

**context** *prob-space*

**begin**

**lemma**  $\text{cdf-distr-measurable [measurable]:}$



**assumes** [measurable]: random-variable borel  $X$   
**shows**  $\text{cdf } (\text{distr } M \text{ borel } X) \in \text{borel-measurable borel}$   
 ⟨proof⟩

**lemma** *cdf-distr-P*:

**assumes** random-variable borel  $X$   
**shows**  $\text{cdf } (\text{distr } M \text{ borel } X) x = \mathcal{P}(\omega \text{ in } M. X \omega \leq x)$   
 ⟨proof⟩

**lemma** *cdf-continuous-distr-P-lt*:

**assumes** random-variable borel  $X$  isCont ( $\text{cdf } (\text{distr } M \text{ borel } X)$ )  $x$   
**shows**  $\text{cdf } (\text{distr } M \text{ borel } X) x = \mathcal{P}(\omega \text{ in } M. X \omega < x)$   
 ⟨proof⟩

**lemma** *cdf-distr-diff-P*:

**assumes**  $x \leq y$   
**and** random-variable borel  $X$   
**shows**  $\text{cdf } (\text{distr } M \text{ borel } X) y - \text{cdf } (\text{distr } M \text{ borel } X) x = \mathcal{P}(\omega \text{ in } M. x < X \omega \wedge X \omega \leq y)$   
 ⟨proof⟩

**lemma** *cdf-distr-max*:

**fixes**  $c::\text{real}$   
**assumes** [measurable]: random-variable borel  $X$   
**shows**  $\text{cdf } (\text{distr } M \text{ borel } (\lambda x. \max (X x) c)) u = \text{cdf } (\text{distr } M \text{ borel } X) u * \text{indicator } \{c..\} u$   
 ⟨proof⟩

**lemma** *cdf-distr-min*:

**fixes**  $c::\text{real}$   
**assumes** [measurable]: random-variable borel  $X$   
**shows**  $\text{cdf } (\text{distr } M \text{ borel } (\lambda x. \min (X x) c)) u = 1 - (1 - \text{cdf } (\text{distr } M \text{ borel } X) u) * \text{indicator } \{..<c\} u$   
 ⟨proof⟩

**lemma** *cdf-distr-floor-P*:

**fixes**  $X :: 'a \Rightarrow \text{real}$   
**assumes** [measurable]: random-variable borel  $X$   
**shows**  $\text{cdf } (\text{distr } M \text{ borel } (\lambda x. \lfloor X x \rfloor)) u = \mathcal{P}(x \text{ in } M. X x < \lfloor u \rfloor + 1)$   
 ⟨proof⟩

**lemma** *expectation-nonneg-tail*:

**assumes** [measurable]: random-variable borel  $X$   
**and**  $X$ -nonneg:  $\bigwedge x. x \in \text{space } M \implies X x \geq 0$   
**defines**  $F u \equiv \text{cdf } (\text{distr } M \text{ borel } X) u$   
**shows**  $(\int^+ x. \text{ennreal } (X x) \partial M) = (\int^+ u \in \{0..\}. \text{ennreal } (1 - F u) \partial \text{lborel})$   
 ⟨proof⟩

**lemma** *expectation-nonpos-tail*:

**assumes** [*measurable*]: *random-variable borel X*  
**and** *X-nonpos*:  $\bigwedge x. x \in \text{space } M \implies X x \leq 0$   
**defines**  $F u \equiv \text{cdf } (\text{distr } M \text{ borel } X) u$   
**shows**  $(\int^{+x}. \text{ennreal } (- X x) \partial M) = (\int^{+u \in \{..0\}}. \text{ennreal } (F u) \partial \text{lborel})$   
*<proof>*

**proposition** *expectation-tail*:

**assumes** [*measurable*]: *integrable M X*  
**defines**  $F u \equiv \text{cdf } (\text{distr } M \text{ borel } X) u$   
**shows**  $\text{expectation } X = (\text{LBINT } u:\{0..\}. 1 - F u) - (\text{LBINT } u:\{..0\}. F u)$   
*<proof>*

**proposition** *distributed-deriv-cdf*:

**assumes** [*measurable*]: *random-variable borel X*  
**defines**  $F u \equiv \text{cdf } (\text{distr } M \text{ borel } X) u$   
**assumes** *finite S*  $\bigwedge x. x \notin S \implies (F \text{ has-real-derivative } f x)$  (*at x*)  
**and** *continuous-on UNIV F f*  $\in \text{borel-measurable lborel}$   
**shows** *distributed M lborel X f*  
*<proof>*

**end**

Define the conditional probability space. This is obtained by rescaling the restricted space of a probability space.

### 3.2 Conditional Probability Space

**lemma** *restrict-prob-space*:

**assumes** *measure-space*  $\Omega A \mu a \in A$   
**and**  $0 < \mu a \mu a < \infty$   
**shows** *prob-space*  $(\text{scale-measure } (1 / \mu a) (\text{restrict-space } (\text{measure-of } \Omega A \mu) a))$   
*<proof>*

**definition** *cond-prob-space* :: '*a measure*  $\implies$  '*a set*  $\implies$  '*a measure* (**infix**  $\langle \_ \rangle$  200)

**where**  $M \downarrow A \equiv \text{scale-measure } (1 / \text{emeasure } M A) (\text{restrict-space } M A)$

**context** *prob-space*

**begin**

**lemma** *cond-prob-space-whole[simp]*:  $M \downarrow \text{space } M = M$

*<proof>*

**lemma** *cond-prob-space-correct*:

**assumes**  $A \in \text{events } \text{prob } A > 0$

**shows** *prob-space*  $(M \downarrow A)$

*<proof>*

**lemma** *space-cond-prob-space*:

**assumes**  $A \in \text{events}$   
**shows**  $\text{space } (M \downarrow A) = A$   
 $\langle \text{proof} \rangle$

**lemma** *sets-cond-prob-space*:  $\text{sets } (M \downarrow A) = (\cap) A \text{ ' events}$   
 $\langle \text{proof} \rangle$

**lemma** *measure-cond-prob-space*:  
**assumes**  $A \in \text{events } B \in \text{events}$   
**and**  $\text{prob } A > 0$   
**shows**  $\text{measure } (M \downarrow A) (B \cap A) = \text{prob } (B \cap A) / \text{prob } A$   
 $\langle \text{proof} \rangle$

**corollary** *measure-cond-prob-space-subset*:  
**assumes**  $A \in \text{events } B \in \text{events } B \subseteq A$   
**and**  $\text{prob } A > 0$   
**shows**  $\text{measure } (M \downarrow A) B = \text{prob } B / \text{prob } A$   
 $\langle \text{proof} \rangle$

**lemma** *cond-cond-prob-space*:  
**assumes**  $A \in \text{events } B \in \text{events } B \subseteq A \text{ prob } B > 0$   
**shows**  $(M \downarrow A) \downarrow B = M \downarrow B$   
 $\langle \text{proof} \rangle$

**lemma** *cond-prob-space-prob*:  
**assumes**  $[\text{measurable}]$ :  $\text{Measurable.pred } M P \text{ Measurable.pred } M Q$   
**and**  $\mathcal{P}(x \text{ in } M. Q x) > 0$   
**shows**  $\text{measure } (M \downarrow \{x \in \text{space } M. Q x\}) \{x \in \text{space } M. P x \wedge Q x\} = \mathcal{P}(x \text{ in } M. P x \mid Q x)$   
 $\langle \text{proof} \rangle$

**lemma** *cond-prob-space-cond-prob*:  
**assumes**  $[\text{measurable}]$ :  $\text{Measurable.pred } M P \text{ Measurable.pred } M Q$   
**and**  $\mathcal{P}(x \text{ in } M. Q x) > 0$   
**shows**  $\mathcal{P}(x \text{ in } M. P x \mid Q x) = \mathcal{P}(x \text{ in } (M \downarrow \{x \in \text{space } M. Q x\}). P x)$   
 $\langle \text{proof} \rangle$

**lemma** *cond-prob-neg*:  
**assumes**  $[\text{measurable}]$ :  $\text{Measurable.pred } M P \text{ Measurable.pred } M Q$   
**and**  $\mathcal{P}(x \text{ in } M. Q x) > 0$   
**shows**  $\mathcal{P}(x \text{ in } M. \neg P x \mid Q x) = 1 - \mathcal{P}(x \text{ in } M. P x \mid Q x)$   
 $\langle \text{proof} \rangle$

**lemma** *random-variable-cond-prob-space*:  
**assumes**  $A \in \text{events } \text{prob } A > 0$   
**and**  $[\text{measurable}]$ :  $\text{random-variable borel } X$   
**shows**  $X \in \text{borel-measurable } (M \downarrow A)$   
 $\langle \text{proof} \rangle$

**lemma** *AE-cond-prob-space-iff*:  
**assumes**  $A \in \text{events}$   $\text{prob } A > 0$   
**shows**  $(AE\ x\ \text{in } M \setminus A. P\ x) \longleftrightarrow (AE\ x\ \text{in } M. x \in A \longrightarrow P\ x)$   
 $\langle \text{proof} \rangle$

**lemma** *integral-cond-prob-space-nn*:  
**assumes**  $A \in \text{events}$   $\text{prob } A > 0$   
**and** *[measurable]: random-variable borel X*  
**and** *nonneg: AE x in M. x ∈ A ⟶ 0 ≤ X x*  
**shows**  $\text{integral}^L (M \setminus A)\ X = \text{expectation } (\lambda x. \text{indicator } A\ x * X\ x) / \text{prob } A$   
 $\langle \text{proof} \rangle$

**end**

Define the complementary cumulative distribution function, also known as tail distribution. The theory presented below is a slight modification of the subsection "Properties of cdf's" in the theory *Distribution-Functions*.

### 3.3 Complementary Cumulative Distribution Function

**definition** *ccdf* ::  $\text{real measure} \Rightarrow \text{real} \Rightarrow \text{real}$   
**where**  $\text{ccdf } M \equiv \lambda x. \text{measure } M\ \{x <..\}$   
— complementary cumulative distribution function (tail distribution)

**lemma** *ccdf-def2*:  $\text{ccdf } M\ x = \text{measure } M\ \{x <..\}$   
 $\langle \text{proof} \rangle$

**context** *finite-borel-measure*  
**begin**

**lemma** *add-cdf-ccdf*:  $\text{cdf } M\ x + \text{ccdf } M\ x = \text{measure } M\ (\text{space } M)$   
 $\langle \text{proof} \rangle$

**lemma** *ccdf-cdf*:  $\text{ccdf } M = (\lambda x. \text{measure } M\ (\text{space } M) - \text{cdf } M\ x)$   
 $\langle \text{proof} \rangle$

**lemma** *cdf-ccdf*:  $\text{cdf } M = (\lambda x. \text{measure } M\ (\text{space } M) - \text{ccdf } M\ x)$   
 $\langle \text{proof} \rangle$

**lemma** *isCont-cdf-ccdf*:  $\text{isCont } (\text{cdf } M)\ x \longleftrightarrow \text{isCont } (\text{ccdf } M)\ x$   
 $\langle \text{proof} \rangle$

**lemma** *isCont-ccdf*:  $\text{isCont } (\text{ccdf } M)\ x \longleftrightarrow \text{measure } M\ \{x\} = 0$   
 $\langle \text{proof} \rangle$

**lemma** *continuous-cdf-ccdf*:  
**shows**  $\text{continuous } F\ (\text{cdf } M) \longleftrightarrow \text{continuous } F\ (\text{ccdf } M)$   
*(is ?LHS ⟷ ?RHS)*  
 $\langle \text{proof} \rangle$

**lemma** *has-real-derivative-cdf-ccdf*:  
 (*cdf M has-real-derivative D*)  $F \longleftrightarrow$  (*ccdf M has-real-derivative -D*)  $F$   
 ⟨*proof*⟩

**lemma** *differentiable-cdf-ccdf*: (*cdf M differentiable F*)  $\longleftrightarrow$  (*ccdf M differentiable F*)  
 ⟨*proof*⟩

**lemma** *deriv-cdf-ccdf*:  
**assumes** *cdf M differentiable at x*  
**shows** *deriv (cdf M) x = - deriv (ccdf M) x*  
 ⟨*proof*⟩

**lemma** *ccdf-diff-eq2*:  
**assumes**  $x \leq y$   
**shows**  $ccdf\ M\ x - cdf\ M\ y = measure\ M\ \{x <..y\}$   
 ⟨*proof*⟩

**lemma** *ccdf-nonincreasing*:  $x \leq y \implies cdf\ M\ x \geq cdf\ M\ y$   
 ⟨*proof*⟩

**lemma** *ccdf-nonneg*:  $ccdf\ M\ x \geq 0$   
 ⟨*proof*⟩

**lemma** *ccdf-bounded*:  $ccdf\ M\ x \leq measure\ M\ (space\ M)$   
 ⟨*proof*⟩

**lemma** *ccdf-lim-at-top*: (*ccdf M*  $\longrightarrow 0$ ) *at-top*  
 ⟨*proof*⟩

**lemma** *ccdf-lim-at-bot*: (*ccdf M*  $\longrightarrow measure\ M\ (space\ M)$ ) *at-bot*  
 ⟨*proof*⟩

**lemma** *ccdf-is-right-cont*: *continuous (at-right a) (ccdf M)*  
 ⟨*proof*⟩

**end**

**context** *prob-space*

**begin**

**lemma** *ccdf-distr-measurable* [*measurable*]:  
**assumes** [*measurable*]: *random-variable borel X*  
**shows**  $ccdf\ (distr\ M\ borel\ X) \in borel\text{-measurable}\ borel$   
 ⟨*proof*⟩

**lemma** *ccdf-distr-P*:  
**assumes** *random-variable borel X*

**shows**  $ccdf (distr M borel X) x = \mathcal{P}(\omega \text{ in } M. X \omega > x)$   
 ⟨proof⟩

**lemma** *ccdf-continuous-distr-P-ge*:

**assumes** *random-variable borel X isCont (ccdf (distr M borel X)) x*  
**shows**  $ccdf (distr M borel X) x = \mathcal{P}(\omega \text{ in } M. X \omega \geq x)$

⟨proof⟩

**lemma** *ccdf-distr-diff-P*:

**assumes**  $x \leq y$

**and** *random-variable borel X*

**shows**  $ccdf (distr M borel X) x - ccdf (distr M borel X) y = \mathcal{P}(\omega \text{ in } M. x < X \omega \wedge X \omega \leq y)$

⟨proof⟩

**end**

**context** *real-distribution*

**begin**

**lemma** *ccdf-bounded-prob*:  $\bigwedge x. ccdf M x \leq 1$

⟨proof⟩

**lemma** *ccdf-lim-at-bot-prob*:  $(ccdf M \longrightarrow 1) \text{ at-bot}$

⟨proof⟩

**end**

Introduce the hazard rate. This notion will be used to define the force of mortality.

### 3.4 Hazard Rate

**context** *prob-space*

**begin**

**definition** *hazard-rate* ::  $('a \Rightarrow real) \Rightarrow real \Rightarrow real$

**where** *hazard-rate X t*  $\equiv$

$Lim (at-right 0) (\lambda dt. \mathcal{P}(x \text{ in } M. t < X x \wedge X x \leq t + dt \mid X x > t) / dt)$

— Here,  $X$  is supposed to be a random variable.

**lemma** *hazard-rate-0-ccdf-0*:

**assumes** [*measurable*]: *random-variable borel X*

**and**  $ccdf (distr M borel X) t = 0$

**shows** *hazard-rate X t = 0*

— Note that division by 0 is calculated as 0 in Isabelle/HOL.

⟨proof⟩

**lemma** *hazard-rate-deriv-cdf*:

**assumes** [measurable]: random-variable borel  $X$   
**and** (cdf (distr  $M$  borel  $X$ )) differentiable at  $t$   
**shows** hazard-rate  $X$   $t = \text{deriv (cdf (distr } M \text{ borel } X)) } t / \text{cdf (distr } M \text{ borel } X)$   
 $t$   
 <proof>

**lemma** deriv-cdf-hazard-rate:  
**assumes** [measurable]: random-variable borel  $X$   
**and** (cdf (distr  $M$  borel  $X$ )) differentiable at  $t$   
**shows** deriv (cdf (distr  $M$  borel  $X$ ))  $t = \text{cdf (distr } M \text{ borel } X) } t * \text{hazard-rate}$   
 $X$   $t$   
 <proof>

**lemma** hazard-rate-deriv-ccdf:  
**assumes** [measurable]: random-variable borel  $X$   
**and** (ccdf (distr  $M$  borel  $X$ )) differentiable at  $t$   
**shows** hazard-rate  $X$   $t = - \text{deriv (ccdf (distr } M \text{ borel } X)) } t / \text{ccdf (distr } M \text{ borel}$   
 $X) } t$   
 <proof>

**lemma** deriv-ccdf-hazard-rate:  
**assumes** [measurable]: random-variable borel  $X$   
**and** (ccdf (distr  $M$  borel  $X$ )) differentiable at  $t$   
**shows** deriv (ccdf (distr  $M$  borel  $X$ ))  $t = - \text{ccdf (distr } M \text{ borel } X) } t * \text{hazard-rate}$   
 $X$   $t$   
 <proof>

**lemma** hazard-rate-deriv-ln-ccdf:  
**assumes** [measurable]: random-variable borel  $X$   
**and** (ccdf (distr  $M$  borel  $X$ )) differentiable at  $t$   
**and** cdf (distr  $M$  borel  $X$ )  $t \neq 0$   
**shows** hazard-rate  $X$   $t = - \text{deriv } (\lambda t. \ln (\text{ccdf (distr } M \text{ borel } X) } t))$   $t$   
 <proof>

**lemma** hazard-rate-has-integral:  
**assumes** [measurable]: random-variable borel  $X$   
**and**  $t \leq u$   
**and** (ccdf (distr  $M$  borel  $X$ )) piecewise-differentiable-on  $\{t < .. < u\}$   
**and** continuous-on  $\{t..u\}$  (cdf (distr  $M$  borel  $X$ ))  
**and**  $\bigwedge s. s \in \{t..u\} \implies \text{cdf (distr } M \text{ borel } X) } s \neq 0$   
**shows**  
 (hazard-rate  $X$  has-integral  $\ln (\text{ccdf (distr } M \text{ borel } X) } t / \text{cdf (distr } M \text{ borel } X)$   
 $u)) \{t..u\}$   
 <proof>

**corollary** hazard-rate-integrable:  
**assumes** [measurable]: random-variable borel  $X$   
**and**  $t \leq u$   
**and** (ccdf (distr  $M$  borel  $X$ )) piecewise-differentiable-on  $\{t < .. < u\}$

**and** *continuous-on*  $\{t..u\}$  (*ccdf* (*distr*  $M$  *borel*  $X$ ))  
**and**  $\bigwedge s. s \in \{t..u\} \implies \text{ccdf} (\text{distr } M \text{ borel } X) s \neq 0$   
**shows** *hazard-rate*  $X$  *integrable-on*  $\{t..u\}$   
*<proof>*

**lemma** *ccdf-exp-cumulative-hazard*:

**assumes** [*measurable*]: *random-variable* *borel*  $X$   
**and**  $t \leq u$   
**and** (*ccdf* (*distr*  $M$  *borel*  $X$ )) *piecewise-differentiable-on*  $\{t < .. < u\}$   
**and** *continuous-on*  $\{t..u\}$  (*ccdf* (*distr*  $M$  *borel*  $X$ ))  
**and**  $\bigwedge s. s \in \{t..u\} \implies \text{ccdf} (\text{distr } M \text{ borel } X) s \neq 0$   
**shows** *ccdf* (*distr*  $M$  *borel*  $X$ )  $u$  / *ccdf* (*distr*  $M$  *borel*  $X$ )  $t =$   
 $\text{exp} (- \text{integral } \{t..u\} (\text{hazard-rate } X))$   
*<proof>*

**lemma** *hazard-rate-density-ccdf*:

**assumes** *distributed*  $M$  *lborel*  $X$   $f$   
**and**  $\bigwedge s. f s \geq 0$   $t < u$  *continuous-on*  $\{t..u\}$   $f$   
**shows** *hazard-rate*  $X$   $t = f t$  / *ccdf* (*distr*  $M$  *borel*  $X$ )  $t$   
*<proof>*

**end**

**end**

**theory** *Interest*

**imports** *Preliminaries*

**begin**

## 4 Theory of Interest

**locale** *interest* =

**fixes**  $i :: \text{real}$  —  $i$  stands for an interest rate.

**assumes** *v-futr-pos*:  $1 + i > 0$  — Assume that the future value is positive.

**begin**

**definition** *i-nom* ::  $\text{nat} \Rightarrow \text{real}$  ( $\langle \$i \{ - \} \rangle$  [0] 200)

**where**  $\$i \{ m \} \equiv m * ((1+i). \wedge (1/m) - 1)$  — nominal interest rate

**definition** *i-force* ::  $\text{real}$  ( $\langle \$\delta \rangle$  200)

**where**  $\$\delta \equiv \ln (1+i)$  — force of interest

**definition** *d-nom* ::  $\text{nat} \Rightarrow \text{real}$  ( $\langle \$d \{ - \} \rangle$  [0] 200)

**where**  $\$d \{ m \} \equiv \$i \{ m \} / (1 + \$i \{ m \} / m)$  — discount rate

**abbreviation** *d-nom-yr* ::  $\text{real}$  ( $\langle \$d \rangle$  200)

**where**  $\$d \equiv \$d \{ 1 \}$  — Post-fix *yr* stands for "year".

**definition** *v-pres* ::  $\text{real}$  ( $\langle \$v \rangle$  200)

**where**  $\$v \equiv 1 / (1+i)$  — present value factor



**definition**  $ann :: nat \Rightarrow nat \Rightarrow real (\langle \$a\{\cdot\}' \rightarrow [0,101] \ 200)$   
**where**  $\$a\{m\}\text{-}n \equiv \sum_{k < n * m}. \$v.\ \tilde{\wedge}((k+1::nat)/m) / m$   
— present value of an immediate annuity

**abbreviation**  $ann\text{-}yr :: nat \Rightarrow real (\langle \$a' \rightarrow [101] \ 200)$   
**where**  $\$a\text{-}n \equiv \$a\{1\}\text{-}n$

**definition**  $acc :: nat \Rightarrow nat \Rightarrow real (\langle \$s\{\cdot\}' \rightarrow [0,101] \ 200)$   
**where**  $\$s\{m\}\text{-}n \equiv \sum_{k < n * m}. (1+i).\ \tilde{\wedge}(k::nat)/m) / m$   
— future value of an immediate annuity  
— The name  $acc$  stands for "accumulation".

**abbreviation**  $acc\text{-}yr :: nat \Rightarrow real (\langle \$s' \rightarrow 200)$   
**where**  $\$s\text{-}n \equiv \$s\{1\}\text{-}n$

**definition**  $ann\text{-}due :: nat \Rightarrow nat \Rightarrow real (\langle \$a'''\{\cdot\}' \rightarrow [0,101] \ 200)$   
**where**  $\$a'''\{m\}\text{-}n \equiv \sum_{k < n * m}. \$v.\ \tilde{\wedge}(k::nat)/m) / m$   
— present value of an annuity-due

**abbreviation**  $ann\text{-}due\text{-}yr :: nat \Rightarrow real (\langle \$a'''' \rightarrow [101] \ 200)$   
**where**  $\$a''\text{-}n \equiv \$a'''\{1\}\text{-}n$

**definition**  $acc\text{-}due :: nat \Rightarrow nat \Rightarrow real (\langle \$s'''\{\cdot\}' \rightarrow [0,101] \ 200)$   
**where**  $\$s'''\{m\}\text{-}n \equiv \sum_{k < n * m}. (1+i).\ \tilde{\wedge}((k+1::nat)/m) / m$   
— future value of an annuity-due

**abbreviation**  $acc\text{-}due\text{-}yr :: nat \Rightarrow real (\langle \$s'''' \rightarrow [101] \ 200)$   
**where**  $\$s''\text{-}n \equiv \$s'''\{1\}\text{-}n$

**definition**  $ann\text{-}cont :: real \Rightarrow real (\langle \$a'''' \rightarrow [101] \ 200)$   
**where**  $\$a'\text{-}n \equiv integral \{0..n\} (\lambda t::real. \$v.\ \tilde{\wedge}t)$   
— present value of a continuous annuity

**definition**  $acc\text{-}cont :: real \Rightarrow real (\langle \$s'''' \rightarrow [101] \ 200)$   
**where**  $\$s'\text{-}n \equiv integral \{0..n\} (\lambda t::real. (1+i).\ \tilde{\wedge}t)$   
— future value of a continuous annuity

**definition**  $perp :: nat \Rightarrow real (\langle \$a\{\cdot\}' \rightarrow [0] \ 200)$   
**where**  $\$a\{m\}\text{-}\infty \equiv 1 / \$i\{m\}$   
— present value of a perpetual annuity

**abbreviation**  $perp\text{-}yr :: real (\langle \$a'\text{-}\infty \rangle 200)$   
**where**  $\$a\text{-}\infty \equiv \$a\{1\}\text{-}\infty$

**definition**  $perp\text{-}due :: nat \Rightarrow real (\langle \$a'''\{\cdot\}' \rightarrow [0] \ 200)$   
**where**  $\$a'''\{m\}\text{-}\infty \equiv 1 / \$d\{m\}$   
— present value of a perpetual annuity-due

**abbreviation** *perp-due-yr* :: real ( $\langle \$a''''-\infty \rangle$  200)  
**where**  $\$a''-\infty \equiv \$a''\{1\}-\infty$

**definition** *ann-incr* :: nat  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  real ( $\langle \$'(I\{-\}a')\{-\}'\rightarrow [0,0,101]$  200)

**where**  $\$(I\{l\}a)\{m\}-n \equiv \sum k < n * m. \$v.\{-(k+1::nat)/m\} * [l*(k+1::nat)/m]$   
/ (l\*m)

- present value of an increasing annuity
- This is my original definition.
- Here, *l* represents the number of increments per unit time.

**abbreviation** *ann-incr-lvl* :: nat  $\Rightarrow$  nat  $\Rightarrow$  real ( $\langle \$'(Ia')\{-\}'\rightarrow [0,101]$  200)

**where**  $\$(Ia)\{m\}-n \equiv \$(I\{1\}a)\{m\}-n$   
— The post-fix *lvl* stands for "level".

**abbreviation** *ann-incr-yr* :: nat  $\Rightarrow$  real ( $\langle \$'(Ia)'\rightarrow [101]$  200)

**where**  $\$(Ia)-n \equiv \$(Ia)\{1\}-n$

**definition** *acc-incr* :: nat  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  real ( $\langle \$'(I\{-\}s')\{-\}'\rightarrow [0,0,101]$  200)

**where**  $\$(I\{l\}s)\{m\}-n \equiv \sum k < n * m. (1+i).\{n-(k+1::nat)/m\} * [l*(k+1::nat)/m]$   
/ (l\*m)

- future value of an increasing annuity

**abbreviation** *acc-incr-lvl* :: nat  $\Rightarrow$  nat  $\Rightarrow$  real ( $\langle \$'(Is')\{-\}'\rightarrow [0,101]$  200)

**where**  $\$(Is)\{m\}-n \equiv \$(I\{1\}s)\{m\}-n$

**abbreviation** *acc-incr-yr* :: nat  $\Rightarrow$  real ( $\langle \$'(Is)'\rightarrow [101]$  200)

**where**  $\$(Is)-n \equiv \$(Is)\{1\}-n$

**definition** *ann-due-incr* :: nat  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  real ( $\langle \$'(I\{-\}a''''')\{-\}'\rightarrow [0,0,101]$  200)

**where**  $\$(I\{l\}a'')\{m\}-n \equiv \sum k < n * m. \$v.\{(k::nat)/m\} * [l*(k+1::nat)/m]$   
/ (l\*m)

**abbreviation** *ann-due-incr-lvl* :: nat  $\Rightarrow$  nat  $\Rightarrow$  real ( $\langle \$'(Ia''''')\{-\}'\rightarrow [0,101]$  200)

**where**  $\$(Ia'')\{m\}-n \equiv \$(I\{1\}a'')\{m\}-n$

**abbreviation** *ann-due-incr-yr* :: nat  $\Rightarrow$  real ( $\langle \$'(Ia''''')'\rightarrow [101]$  200)

**where**  $\$(Ia'')-n \equiv \$(Ia'')\{1\}-n$

**definition** *acc-due-incr* :: nat  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  real ( $\langle \$'(I\{-\}s''''')\{-\}'\rightarrow [0,0,101]$  200)

**where**  $\$(I\{l\}s'')\{m\}-n \equiv \sum k < n * m. (1+i).\{n-(k::nat)/m\} * [l*(k+1::nat)/m]$   
/ (l\*m)

**abbreviation** *acc-due-incr-lvl* :: nat  $\Rightarrow$  nat  $\Rightarrow$  real ( $\langle \$'(Is''''')\{-\}'\rightarrow [0,101]$  200)

**where**  $\$(Is'')\{m\}-n \equiv \$(I\{1\}s'')\{m\}-n$

**abbreviation**  $acc\text{-}due\text{-}incr\text{-}yr :: nat \Rightarrow real \langle \$(Is''''')' \rightarrow [101] 200 \rangle$   
**where**  $\$(Is'')\text{-}n \equiv \$(Is') \wedge \{1\}\text{-}n$

**definition**  $perp\text{-}incr :: nat \Rightarrow nat \Rightarrow real \langle \$(I \wedge \{-\})a' \wedge \{-\}'\text{-}\infty \rangle [0,0] 200$   
**where**  $\$(I \wedge \{l\}a) \wedge \{m\}\text{-}\infty \equiv \lim (\lambda n. \$(I \wedge \{l\}a) \wedge \{m\}\text{-}n)$

**abbreviation**  $perp\text{-}incr\text{-}lvl :: nat \Rightarrow real \langle \$(Ia') \wedge \{-\}'\text{-}\infty \rangle [0] 200$   
**where**  $\$(Ia) \wedge \{m\}\text{-}\infty \equiv \$(I \wedge \{1\}a) \wedge \{m\}\text{-}\infty$

**abbreviation**  $perp\text{-}incr\text{-}yr :: real \langle \$(Ia')'\text{-}\infty \rangle 200$   
**where**  $\$(Ia)\text{-}\infty \equiv \$(Ia) \wedge \{1\}\text{-}\infty$

**definition**  $perp\text{-}due\text{-}incr :: nat \Rightarrow nat \Rightarrow real \langle \$(I \wedge \{-\})a'''' \wedge \{-\}'\text{-}\infty \rangle [0,0] 200$   
**where**  $\$(I \wedge \{l\}a'') \wedge \{m\}\text{-}\infty \equiv \lim (\lambda n. \$(I \wedge \{l\}a'') \wedge \{m\}\text{-}n)$

**abbreviation**  $perp\text{-}due\text{-}incr\text{-}lvl :: nat \Rightarrow real \langle \$(Ia''''') \wedge \{-\}'\text{-}\infty \rangle [0] 200$   
**where**  $\$(Ia'') \wedge \{m\}\text{-}\infty \equiv \$(I \wedge \{1\}a'') \wedge \{m\}\text{-}\infty$

**abbreviation**  $perp\text{-}due\text{-}incr\text{-}yr :: real \langle \$(Ia''''')'\text{-}\infty \rangle 200$   
**where**  $\$(Ia'')\text{-}\infty \equiv \$(Ia'') \wedge \{1\}\text{-}\infty$

**lemma**  $v\text{-}futr\text{-}m\text{-}pos: 1 + \$(i \wedge \{m\})/m > 0$  **if**  $m \neq 0$  **for**  $m::nat$   
 $\langle proof \rangle$

**lemma**  $i\text{-}nom\text{-}1[simp]: \$(i \wedge \{1\}) = i$   
 $\langle proof \rangle$

**lemma**  $i\text{-}nom\text{-}eff: (1 + \$(i \wedge \{m\})/m) \wedge m = 1 + i$  **if**  $m \neq 0$  **for**  $m::nat$   
 $\langle proof \rangle$

**lemma**  $i\text{-}nom\text{-}i: 1 + \$(i \wedge \{m\})/m = (1+i) \wedge (1/m)$  **if**  $m \neq 0$  **for**  $m::nat$   
 $\langle proof \rangle$

**lemma**  $i\text{-}nom\text{-}0\text{-}iff\text{-}i\text{-}0: \$(i \wedge \{m\}) = 0 \longleftrightarrow i = 0$  **if**  $m \neq 0$  **for**  $m::nat$   
 $\langle proof \rangle$

**lemma**  $i\text{-}nom\text{-}pos\text{-}iff\text{-}i\text{-}pos: \$(i \wedge \{m\}) > 0 \longleftrightarrow i > 0$  **if**  $m \neq 0$  **for**  $m::nat$   
 $\langle proof \rangle$

**lemma**  $e\text{-}delta: exp (\$ \delta) = 1 + i$   
 $\langle proof \rangle$

**lemma**  $delta\text{-}0\text{-}iff\text{-}i\text{-}0: \$ \delta = 0 \longleftrightarrow i = 0$   
 $\langle proof \rangle$

**lemma**  $lim\text{-}i\text{-}nom: (\lambda m. \$(i \wedge \{m\})) \longrightarrow \$ \delta$   
 $\langle proof \rangle$

**lemma**  $d\text{-}nom\text{-}0\text{-}iff\text{-}i\text{-}0: \$ d \wedge \{m\} = 0 \longleftrightarrow i = 0$  **if**  $m \neq 0$  **for**  $m::nat$

*<proof>*

**lemma** *d-nom-pos-iff-i-pos*:  $\$d^{\wedge}\{m\} > 0 \iff i > 0$  **if**  $m \neq 0$  **for**  $m::nat$   
*<proof>*

**lemma** *d-nom-i-nom*:  $1 - \$d^{\wedge}\{m\}/m = 1 / (1 + \$i^{\wedge}\{m\}/m)$  **if**  $m \neq 0$  **for**  $m::nat$   
*<proof>*

**lemma** *lim-d-nom*:  $(\lambda m. \$d^{\wedge}\{m\}) \longrightarrow \$\delta$   
*<proof>*

**lemma** *v-pos*:  $\$v > 0$   
*<proof>*

**lemma** *v-1-iff-i-0*:  $\$v = 1 \iff i = 0$   
*<proof>*

**lemma** *v-lt-1-iff-i-pos*:  $\$v < 1 \iff i > 0$   
*<proof>*

**lemma** *v-i-nom*:  $\$v = (1 + \$i^{\wedge}\{m\}/m)^{\wedge{-}m}$  **if**  $m \neq 0$  **for**  $m::nat$   
*<proof>*

**lemma** *i-v*:  $1 + i = \$v^{\wedge{-}1}$   
*<proof>*

**lemma** *i-v-powr*:  $(1 + i)^{\wedge{a}} = \$v^{\wedge{-}a}$  **for**  $a::real$   
*<proof>*

**lemma** *v-delta*:  $\ln (\$v) = - \$\delta$   
*<proof>*

**lemma** *is-derive-vpow*: *DERIV*  $(\lambda t. \$v^{\wedge}t) t :> - \$\delta * \$v^{\wedge}t$   
*<proof>*

**lemma** *d-nom-v*:  $\$d^{\wedge}\{m\} = m * (1 - \$v^{\wedge}(1/m))$  **if**  $m \neq 0$  **for**  $m::nat$   
*<proof>*

**lemma** *d-nom-i-nom-v*:  $\$d^{\wedge}\{m\} = \$i^{\wedge}\{m\} * \$v^{\wedge}(1/m)$  **if**  $m \neq 0$  **for**  $m::nat$   
*<proof>*

**lemma** *a-calc*:  $\$a^{\wedge}\{m\}-n = (1 - \$v^{\wedge}n) / \$i^{\wedge}\{m\}$  **if**  $m \neq 0$   $i \neq 0$  **for**  $n m :: nat$   
*<proof>*

**lemma** *a-calc-i-0*:  $\$a^{\wedge}\{m\}-n = n$  **if**  $m \neq 0$   $i = 0$  **for**  $n m :: nat$   
*<proof>*

**lemma** *s-calc-i-0*:  $\$s^{\wedge}\{m\}-n = n$  **if**  $m \neq 0$   $i = 0$  **for**  $n m :: nat$   
*<proof>*

**lemma**  $s$ - $a$ :  $\$s^{\wedge}\{m\}$ - $n = (1+i)^{\wedge}n * \$a^{\wedge}\{m\}$ - $n$  **if**  $m \neq 0$  **for**  $n m :: nat$   
 ⟨proof⟩

**lemma**  $s$ - $calc$ :  $\$s^{\wedge}\{m\}$ - $n = ((1+i)^{\wedge}n - 1) / \$i^{\wedge}\{m\}$  **if**  $m \neq 0 i \neq 0$  **for**  $n m :: nat$   
 ⟨proof⟩

**lemma**  $a''$ - $a$ :  $\$a''^{\wedge}\{m\}$ - $n = (1+i)^{\wedge}(1/m) * \$a^{\wedge}\{m\}$ - $n$  **if**  $m \neq 0$  **for**  $m::nat$   
 ⟨proof⟩

**lemma**  $a$ - $a''$ :  $\$a^{\wedge}\{m\}$ - $n = \$v.^{\wedge}(1/m) * \$a''^{\wedge}\{m\}$ - $n$  **if**  $m \neq 0$  **for**  $m::nat$   
 ⟨proof⟩

**lemma**  $a''$ - $calc$ - $i$ - $0$ :  $\$a''^{\wedge}\{m\}$ - $n = n$  **if**  $m \neq 0 i = 0$  **for**  $n m :: nat$   
 ⟨proof⟩

**lemma**  $s''$ - $calc$ - $i$ - $0$ :  $\$s''^{\wedge}\{m\}$ - $n = n$  **if**  $m \neq 0 i = 0$  **for**  $n m :: nat$   
 ⟨proof⟩

**lemma**  $a''$ - $calc$ :  $\$a''^{\wedge}\{m\}$ - $n = (1 - \$v^{\wedge}n) / \$d^{\wedge}\{m\}$  **if**  $m \neq 0 i \neq 0$  **for**  $n m :: nat$   
 ⟨proof⟩

**lemma**  $s''$ - $s$ :  $\$s''^{\wedge}\{m\}$ - $n = (1+i)^{\wedge}(1/m) * \$s^{\wedge}\{m\}$ - $n$  **if**  $m \neq 0$  **for**  $m::nat$   
 ⟨proof⟩

**lemma**  $s$ - $s''$ :  $\$s^{\wedge}\{m\}$ - $n = \$v.^{\wedge}(1/m) * \$s''^{\wedge}\{m\}$ - $n$  **if**  $m \neq 0$  **for**  $m::nat$   
 ⟨proof⟩

**lemma**  $s''$ - $calc$ :  $\$s''^{\wedge}\{m\}$ - $n = ((1+i)^{\wedge}n - 1) / \$d^{\wedge}\{m\}$  **if**  $m \neq 0 i \neq 0$  **for**  $n m :: nat$   
 ⟨proof⟩

**lemma**  $s''$ - $a''$ :  $\$s''^{\wedge}\{m\}$ - $n = (1+i)^{\wedge}n * \$a''^{\wedge}\{m\}$ - $n$  **if**  $m \neq 0$  **for**  $m::nat$   
 ⟨proof⟩

**lemma**  $a'$ - $calc$ :  $\$a'^{-}n = (1 - \$v.^{\wedge}n) / \$\delta$  **if**  $i \neq 0 n \geq 0$  **for**  $n::real$   
 ⟨proof⟩

**lemma**  $a'$ - $calc$ - $i$ - $0$ :  $\$a'^{-}n = n$  **if**  $i = 0 n \geq 0$  **for**  $n::real$   
 ⟨proof⟩

**lemma**  $s'$ - $calc$ :  $\$s'^{-}n = ((1+i)^{\wedge}n - 1) / \$\delta$  **if**  $i \neq 0 n \geq 0$  **for**  $n::real$   
 ⟨proof⟩

**lemma**  $s'$ - $calc$ - $i$ - $0$ :  $\$s'^{-}n = n$  **if**  $i = 0 n \geq 0$  **for**  $n::real$   
 ⟨proof⟩

**lemma**  $s'-a'$ :  $\$(s'-n) = (1+i).\hat{\ }n * \$a'-n$  if  $n \geq 0$  for  $n::real$   
 ⟨proof⟩

**lemma**  $lim-m-a$ :  $(\lambda m. \$a\hat{\ }m-n) \longrightarrow \$a'-n$  for  $n::nat$   
 ⟨proof⟩

**lemma**  $lim-m-a''$ :  $(\lambda m. \$a''\hat{\ }m-n) \longrightarrow \$a'-n$  for  $n::nat$   
 ⟨proof⟩

**lemma**  $lim-m-s$ :  $(\lambda m. \$s\hat{\ }m-n) \longrightarrow \$s'-n$  for  $n::nat$   
 ⟨proof⟩

**lemma**  $lim-m-s''$ :  $(\lambda m. \$s''\hat{\ }m-n) \longrightarrow \$s'-n$  for  $n::nat$   
 ⟨proof⟩

**lemma**  $lim-n-a$ :  $(\lambda n. \$a\hat{\ }m-n) \longrightarrow \$a\hat{\ }m-\infty$  if  $m \neq 0$   $i > 0$  for  $m::nat$   
 ⟨proof⟩

**lemma**  $lim-n-a''$ :  $(\lambda n. \$a''\hat{\ }m-n) \longrightarrow \$a''\hat{\ }m-\infty$  if  $m \neq 0$   $i > 0$  for  $m::nat$   
 ⟨proof⟩

**lemma**  $lsm-lam$ :  $\$(I\hat{\ }l)s\hat{\ }m-n = (1+i).\hat{\ }n * \$(I\hat{\ }l)a\hat{\ }m-n$   
 if  $l \neq 0$   $m \neq 0$  for  $l$   $n$   $m :: nat$   
 ⟨proof⟩

**lemma**  $Iam-calc$ :  $\$(Ia)\hat{\ }m-n = (\sum j < n. (j+1)/m * (\sum k=j*m..<(j+1)*m. \$v.\hat{\ }((k+1)/m)))$   
 if  $m \neq 0$  for  $n$   $m :: nat$   
 ⟨proof⟩

**lemma**  $Ism-calc$ :  $\$(Is)\hat{\ }m-n = (\sum j < n. (j+1)/m * (\sum k=j*m..<(j+1)*m. (1+i).\hat{\ }((n-(k+1)/m)))$   
 if  $m \neq 0$  for  $n$   $m :: nat$   
 ⟨proof⟩

**lemma**  $Imam-calc-aux$ :  $\$(I\hat{\ }m)a\hat{\ }m-n = (\sum k < n*m. \$v.\hat{\ }((k+1)/m) * (k+1) / m^2)$   
 if  $m \neq 0$  for  $m::nat$   
 ⟨proof⟩

**lemma**  $Imam-calc$ :  
 $\$(I\hat{\ }m)a\hat{\ }m-n = (\$v.\hat{\ }1/m) * (1 - (n*m+1)*\$v.\hat{\ }n + n*m*\$v.\hat{\ }((n+1)/m)) / (m*(1-\$v.\hat{\ }1/m))^2$   
 if  $i \neq 0$   $m \neq 0$  for  $n$   $m :: nat$   
 ⟨proof⟩

**lemma**  $Imam-calc-i-0$ :  $\$(I\hat{\ }m)a\hat{\ }m-n = (n*m+1)*n / (2*m)$  if  $i = 0$   $m \neq 0$   
 for  $n$   $m :: nat$   
 ⟨proof⟩

**lemma**  $Imsm-calc$ :

$$\$(I\{m\}s)\{m\}-n = ((1+i).\wedge(n+1/m) - (n*m+1)*(1+i).\wedge(1/m) + n*m) / (m*((1+i).\wedge(1/m)-1))\wedge 2$$
**if**  $i \neq 0$   $m \neq 0$  **for**  $n$   $m :: \text{nat}$   
 <proof>

**lemma** *Imsm-calc-i-0*:  $\$(I\{m\}s)\{m\}-n = (n*m+1)*n / (2*m)$  **if**  $i = 0$   $m \neq 0$   
**for**  $n$   $m :: \text{nat}$   
 <proof>

**lemma** *Ila''m-Ilam*:  $\$(I\{l\}a'')\{m\}-n = (1+i).\wedge(1/m) * \$(I\{l\}a)\{m\}-n$   
**if**  $l \neq 0$   $m \neq 0$  **for**  $l$   $m$   $n :: \text{nat}$   
 <proof>

**lemma** *Ia''m-calc*:  $\$(Ia'')\{m\}-n = (\sum j < n. (j+1)/m * (\sum k=j*m..<(j+1)*m. \$v.\wedge(k/m)))$   
**if**  $m \neq 0$  **for**  $n$   $m :: \text{nat}$   
 <proof>

**lemma** *Ima''m-calc-aux*:  $\$(I\{m\}a'')\{m\}-n = (\sum k < n*m. \$v.\wedge(k/m) * (k+1) / m\wedge 2)$   
**if**  $m \neq 0$  **for**  $m :: \text{nat}$   
 <proof>

**lemma** *Ima''m-calc*:  $\$(I\{m\}a'')\{m\}-n = (1 - (n*m+1)*\$v.\wedge n + n*m*\$v.\wedge(n+1/m)) / (m*(1-\$v.\wedge(1/m)))\wedge 2$   
**if**  $i \neq 0$   $m \neq 0$  **for**  $n$   $m :: \text{nat}$   
 <proof>

**lemma** *Ils''m-Ilsm*:  $\$(I\{l\}s'')\{m\}-n = (1+i).\wedge(1/m) * \$(I\{l\}s)\{m\}-n$   
**if**  $l \neq 0$   $m \neq 0$  **for**  $l$   $m$   $n :: \text{nat}$   
 <proof>

**lemma** *Ims''m-calc*:  

$$\$(I\{m\}s'')\{m\}-n = (1+i).\wedge(1/m) * ((1+i).\wedge(n+1/m) - (n*m+1)*(1+i).\wedge(1/m) + n*m) / (m*((1+i).\wedge(1/m)-1))\wedge 2$$
**if**  $i \neq 0$   $m \neq 0$  **for**  $n$   $m :: \text{nat}$   
 <proof>

**lemma** *lim-Imam*:  $(\lambda n. \$(I\{m\}a)\{m\}-n) \longrightarrow 1 / (\$i\{m\}*\$d\{m\})$  **if**  $m \neq 0$   $i > 0$  **for**  $m :: \text{nat}$   
 <proof>

**lemma** *perp-incr-calc*:  $\$(I\{m\}a)\{m\}-\infty = 1 / (\$i\{m\}*\$d\{m\})$  **if**  $m \neq 0$   $i > 0$  **for**  $m :: \text{nat}$   
 <proof>

**lemma** *lim-Ima''m*:  $(\lambda n. \$(I\{m\}a'')\{m\}-n) \longrightarrow 1 / (\$d\{m\})\wedge 2$  **if**  $m \neq 0$   $i > 0$  **for**  $m :: \text{nat}$

*<proof>*

**lemma** *perp-due-incr-calc*:  $(I^{\wedge\{m\}a'})^{\wedge\{m\}-\infty} = 1 / (d^{\wedge\{m\}})^{\wedge 2}$  **if**  $m \neq 0$  **and**  $i > 0$  **for**  $m::nat$   
*<proof>*

**end**

**end**

**theory** *Survival-Model*

**imports** *HOL-Library.Rewrite* *HOL-Library.Extended-Nonnegative-Real* *HOL-Library.Extended-Real*  
*HOL-Probability.Probability Preliminaries*

**begin**

## 5 Survival Model

The survival model is built on the probability space  $\mathfrak{M}$ . Additionally, the random variable  $X : space\ \mathfrak{M} \rightarrow \mathbb{R}$  is introduced, which represents the age at death.

**locale** *prob-space-actuary = MM-PS: prob-space*  $\mathfrak{M}$  **for**  $\mathfrak{M}$

— Since the letter M may be used as a commutation function, adopt the letter  $\mathfrak{M}$  instead as a notation for the measure space.

**locale** *survival-model = prob-space-actuary +*

**fixes**  $X :: 'a \Rightarrow real$

**assumes** *X-RV[simp]*: *MM-PS.random-variable* (*borel*  $:: real\ measure$ )  $X$

**and** *X-pos-AE[simp]*: *AE*  $\xi$  *in*  $\mathfrak{M}$ .  $X\ \xi > 0$

**begin**

### 5.1 General Theory of Survival Model

**interpretation** *distrX-RD: real-distribution distr*  $\mathfrak{M}$  *borel*  $X$

*<proof>*

**lemma** *X-le-event[simp]*:  $\{\xi \in space\ \mathfrak{M}. X\ \xi \leq x\} \in MM-PS.events$

*<proof>*

**lemma** *X-gt-event[simp]*:  $\{\xi \in space\ \mathfrak{M}. X\ \xi > x\} \in MM-PS.events$

*<proof>*

**lemma** *X-compl-le-gt*:  $space\ \mathfrak{M} - \{\xi \in space\ \mathfrak{M}. X\ \xi \leq x\} = \{\xi \in space\ \mathfrak{M}. X\ \xi > x\}$  **for**  $x::real$

*<proof>*

**lemma** *X-compl-gt-le*:  $space\ \mathfrak{M} - \{\xi \in space\ \mathfrak{M}. X\ \xi > x\} = \{\xi \in space\ \mathfrak{M}. X\ \xi \leq x\}$  **for**  $x::real$

*<proof>*



### 5.1.1 Introduction of Survival Function for $X$

Note that  $ccdf$  ( $distr \mathfrak{M}$  borel  $X$ ) is the survival (distributive) function for  $X$ .

**lemma**  $ccdfX-0-1$ :  $ccdf$  ( $distr \mathfrak{M}$  borel  $X$ )  $0 = 1$   
 $\langle proof \rangle$

**lemma**  $ccdfX-unborn-1$ :  $ccdf$  ( $distr \mathfrak{M}$  borel  $X$ )  $x = 1$  if  $x \leq 0$   
 $\langle proof \rangle$

**definition**  $death-pt$  ::  $ereal$  ( $\langle \$\psi \rangle$ )

**where**  $\$ \psi \equiv Inf$  ( $ereal$  '  $\{x \in \mathbb{R}. ccdf$  ( $distr \mathfrak{M}$  borel  $X$ )  $x = 0\}$ )

— This is my original notation, which is used to develop life insurance mathematics rigorously.

**lemma**  $psi-nonneg$ :  $\$ \psi \geq 0$   
 $\langle proof \rangle$

**lemma**  $ccdfX-beyond-0$ :  $ccdf$  ( $distr \mathfrak{M}$  borel  $X$ )  $x = 0$  if  $x > \$ \psi$  for  $x::real$   
 $\langle proof \rangle$

**lemma**  $ccdfX-psi-0$ :  $ccdf$  ( $distr \mathfrak{M}$  borel  $X$ ) ( $real-of-ereal$   $\$ \psi$ ) =  $0$  if  $\$ \psi < \infty$   
 $\langle proof \rangle$

**lemma**  $ccdfX-0-equiv$ :  $ccdf$  ( $distr \mathfrak{M}$  borel  $X$ )  $x = 0 \iff x \geq \$ \psi$  for  $x::real$   
 $\langle proof \rangle$

**lemma**  $psi-pos[simp]$ :  $\$ \psi > 0$   
 $\langle proof \rangle$

**corollary**  $psi-pos'[simp]$ :  $\$ \psi > ereal 0$   
 $\langle proof \rangle$

### 5.1.2 Introduction of Future-Lifetime Random Variable $T(x)$

**definition**  $alive$  ::  $real \Rightarrow 'a$  set

**where**  $alive$   $x \equiv \{\xi \in space \mathfrak{M}. X \xi > x\}$

**lemma**  $alive-event[simp]$ :  $alive$   $x \in MM-PS.events$  for  $x::real$   
 $\langle proof \rangle$

**lemma**  $X-alivex-measurable[measurable, simp]$ :  $X \in borel-measurable$  ( $\mathfrak{M} \downarrow alive$   $x$ ) for  $x::real$   
 $\langle proof \rangle$

**definition**  $futr-life$  ::  $real \Rightarrow ('a \Rightarrow real)$  ( $\langle T \rangle$ )

**where**  $T$   $x \equiv (\lambda \xi. X \xi - x)$

— Note that  $T(x) : space \mathfrak{M} \rightarrow \mathbb{R}$  represents the time until death of a person aged  $x$ .

**lemma**  $T0\text{-eq-}X[simp]$ :  $T\ 0 = X$   
 ⟨proof⟩

**lemma**  $Tx\text{-measurable}[measurable, simp]$ :  $T\ x \in \text{borel-measurable } \mathfrak{M}$  for  $x::real$   
 ⟨proof⟩

**lemma**  $Tx\text{-alivex-measurable}[measurable, simp]$ :  $T\ x \in \text{borel-measurable } (\mathfrak{M} \mid \text{alive } x)$  for  $x::real$   
 ⟨proof⟩

**lemma**  $\text{alive-}T$ :  $\text{alive } x = \{\xi \in \text{space } \mathfrak{M}. T\ x\ \xi > 0\}$  for  $x::real$   
 ⟨proof⟩

**lemma**  $\text{alivex-}Tx\text{-pos}[simp]$ :  $0 < T\ x\ \xi$  if  $\xi \in \text{space } (\mathfrak{M} \mid \text{alive } x)$  for  $x::real$   
 ⟨proof⟩

**lemma**  $PT0\text{-eq-}PX\text{-lborel}$ :  $\mathcal{P}(\xi \text{ in } \mathfrak{M}. T\ 0\ \xi \in A \mid T\ 0\ \xi > 0) = \mathcal{P}(\xi \text{ in } \mathfrak{M}. X\ \xi \in A)$   
 if  $A \in \text{sets lborel}$  for  $A :: \text{real set}$   
 ⟨proof⟩

### 5.1.3 Actuarial Notations on the Survival Model

**definition**  $\text{survive} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  ( $\langle \$p'\text{-}\{\&x\}\rangle [0,0] 200$ )

**where**  $\$p\text{-}\{t\&x\} \equiv \text{ccdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T\ x))\ t$

— the probability that a person aged  $x$  will survive for  $t$  years

— Note that the function  $\$p\text{-}\{\cdot\&x\}$  is the survival function on  $(\mathfrak{M} \mid \text{alive } x)$  for the random variable  $T(x)$ .

— The parameter  $t$  is usually nonnegative, but theoretically it can be negative.

**abbreviation**  $\text{survive-1} :: \text{real} \Rightarrow \text{real}$  ( $\langle \$p'\text{-}\rightarrow [101] 200$ )

**where**  $\$p\text{-}x \equiv \$p\text{-}\{1\&x\}$

**definition**  $\text{die} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  ( $\langle \$q'\text{-}\{\&x\}\rangle [0,0] 200$ )

**where**  $\$q\text{-}\{t\&x\} \equiv \text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T\ x))\ t$

— the probability that a person aged  $x$  will die within  $t$  years

— Note that the function  $\$q\text{-}\{\cdot\&x\}$  is the cumulative distributive function on  $(\mathfrak{M} \mid \text{alive } x)$  for the random variable  $T(x)$ .

— The parameter  $t$  is usually nonnegative, but theoretically it can be negative.

**abbreviation**  $\text{die-1} :: \text{real} \Rightarrow \text{real}$  ( $\langle \$q'\text{-}\rightarrow [101] 200$ )

**where**  $\$q\text{-}x \equiv \$q\text{-}\{1\&x\}$

**definition**  $\text{die-defer} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow \text{real}$  ( $\langle \$q'\text{-}\{\text{-}\{\&x\}\}\rangle [0,0,0] 200$ )

**where**  $\$q\text{-}\{f\mid t\&x\} = |\$q\text{-}\{f+t\&x\} - \$q\text{-}\{f\&x\}|$

— the probability that a person aged  $x$  will die within  $t$  years, deferred  $f$  years

— The parameters  $f$  and  $t$  are usually nonnegative, but theoretically they can be negative.

**abbreviation** *die-defer-1* ::  $real \Rightarrow real \Rightarrow real$  ( $\langle \$q'\{-|\&- \} \rangle [0,0] 200$ )

**where**  $\$q\{-f|\&x\} \equiv \$q\{-f|1\&x\}$

**definition** *life-expect* ::  $real \Rightarrow real$  ( $\langle \$e'\circ'\{- \} \rangle [101] 200$ )

**where**  $\$e'\circ'\{-x\} \equiv integral^L (\mathfrak{M} \downarrow alive\ x) (T\ x)$

— complete life expectation

— Note that  $\$e'\circ'\{-x\}$  is calculated as  $0$  when  $nn\text{-integral} (\mathfrak{M} \downarrow alive\ x) (T\ x) = \infty$ .

**definition** *temp-life-expect* ::  $real \Rightarrow real \Rightarrow real$  ( $\langle \$e'\circ'\{-\cdot-\} \rangle [0,0] 200$ )

**where**  $\$e'\circ'\{-x:n\} \equiv integral^L (\mathfrak{M} \downarrow alive\ x) (\lambda\xi. \min (T\ x\ \xi)\ n)$

— temporary complete life expectation

**definition** *curt-life-expect* ::  $real \Rightarrow real$  ( $\langle \$e'\{- \} \rangle [101] 200$ )

**where**  $\$e\{-x\} \equiv integral^L (\mathfrak{M} \downarrow alive\ x) (\lambda\xi. \lfloor T\ x\ \xi \rfloor)$

— curtate life expectation

— Note that  $\$e\{-x\}$  is calculated as  $0$  when  $nn\text{-integral} (\mathfrak{M} \downarrow alive\ x) \lfloor T\ x \rfloor = \infty$ .

**definition** *temp-curt-life-expect* ::  $real \Rightarrow real \Rightarrow real$  ( $\langle \$e'\{-\cdot-\} \rangle [0,0] 200$ )

**where**  $\$e\{-x:n\} \equiv integral^L (\mathfrak{M} \downarrow alive\ x) (\lambda\xi. \lfloor \min (T\ x\ \xi)\ n \rfloor)$

— temporary curtate life expectation

— In the definition  $n$  can be a real number, but in practice  $n$  is usually a natural number.

#### 5.1.4 Properties of Survival Function for $T(x)$

**context**

**fixes**  $x::real$

**assumes**  $x\text{-lt-psi}[simp]: x < \psi$

**begin**

**lemma** *PXx-pos[simp]*:  $\mathcal{P}(\xi \text{ in } \mathfrak{M}. X\ \xi > x) > 0$

$\langle proof \rangle$

**lemma** *PTx-pos[simp]*:  $\mathcal{P}(\xi \text{ in } \mathfrak{M}. T\ x\ \xi > 0) > 0$

$\langle proof \rangle$

**interpretation** *alivex-PS*:  $prob\text{-space } \mathfrak{M} \downarrow alive\ x$

$\langle proof \rangle$

**interpretation** *distrTx-RD*:  $real\text{-distribution } distr (\mathfrak{M} \downarrow alive\ x) \text{ borel } (T\ x) \langle proof \rangle$

**lemma** *ccdfTx-cond-prob*:

$ccdf (distr (\mathfrak{M} \downarrow alive\ x) \text{ borel } (T\ x))\ t = \mathcal{P}(\xi \text{ in } \mathfrak{M}. T\ x\ \xi > t \mid T\ x\ \xi > 0)$  **for**

$t::real$

$\langle proof \rangle$

**lemma** *ccdfTx-0-1*:  $ccdf (distr (\mathfrak{M} \downarrow alive\ x) \text{ borel } (T\ x))\ 0 = 1$

*<proof>*

**lemma** *ccdfTx-nonpos-1*:  $ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)) t = 1$  **if**  $t \leq 0$  **for**  
 $t :: \text{real}$   
*<proof>*

**lemma** *ccdfTx-0-equiv*:  $ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)) t = 0 \iff x+t \geq$   
 $\psi$  **for**  $t :: \text{real}$   
*<proof>*

**lemma** *ccdfTx-continuous-on-nonpos[simp]*:  
 $continuous\text{-on } \{..0\} (ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)))$   
*<proof>*

**lemma** *ccdfTx-differentiable-on-nonpos[simp]*:  
 $(ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x))) \text{ differentiable-on } \{..0\}$   
*<proof>*

**lemma** *ccdfTx-has-real-derivative-0-at-neg*:  
 $(ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)) \text{ has-real-derivative } 0) (at t)$  **if**  $t < 0$  **for**  
 $t :: \text{real}$   
*<proof>*

**lemma** *ccdfTx-integrable-Icc*:  
 $set\text{-integrable } lborel \{a..b\} (ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)))$  **for**  $a b :: \text{real}$   
*<proof>*

**corollary** *ccdfTx-integrable-on-Icc*:  
 $ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)) \text{ integrable-on } \{a..b\}$  **for**  $a b :: \text{real}$   
*<proof>*

**lemma** *ccdfTx-PX*:  
 $ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)) t = \mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x+t) / \mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x)$   
**if**  $t \geq 0$  **for**  $t :: \text{real}$   
*<proof>*

**lemma** *ccdfTx-ccdfX*:  $ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)) t =$   
 $ccdf (distr \mathfrak{M} \text{ borel } X) (x + t) / ccdf (distr \mathfrak{M} \text{ borel } X) x$   
**if**  $t \geq 0$  **for**  $t :: \text{real}$   
*<proof>*

**lemma** *ccdfT0-eq-ccdfX*:  $ccdf (distr (\mathfrak{M} \downarrow \text{alive } 0) \text{ borel } (T 0)) = ccdf (distr \mathfrak{M} \text{ borel } X)$   
*<proof>*

**lemma** *continuous-ccdfX-ccdfTx*:  
 $continuous (at (x+t) \text{ within } \{x..\}) (ccdf (distr \mathfrak{M} \text{ borel } X)) \iff$   
 $continuous (at t \text{ within } \{0..\}) (ccdf (distr (\mathfrak{M} \downarrow \text{alive } x) \text{ borel } (T x)))$

**if  $t \geq 0$  for  $t::real$**   
 $\langle proof \rangle$

**lemma isCont-ccdfX-ccdfTx:**  
 $isCont (ccdf (distr \mathfrak{M} borel X)) (x+t) \longleftrightarrow$   
 $isCont (ccdf (distr (\mathfrak{M} \downarrow alive x) borel (T x))) t$   
**if  $t > 0$  for  $t::real$**   
 $\langle proof \rangle$

**lemma has-real-derivative-ccdfX-ccdfTx:**  
 $((ccdf (distr \mathfrak{M} borel X)) has-real-derivative D) (at (x+t)) \longleftrightarrow$   
 $((ccdf (distr (\mathfrak{M} \downarrow alive x) borel (T x))) has-real-derivative (D / \mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x))) (at t)$   
**if  $t > 0$  for  $t D :: real$**   
 $\langle proof \rangle$

**lemma differentiable-ccdfX-ccdfTx:**  
 $(ccdf (distr \mathfrak{M} borel X)) \text{ differentiable at } (x+t) \longleftrightarrow$   
 $(ccdf (distr (\mathfrak{M} \downarrow alive x) borel (T x))) \text{ differentiable at } t$   
**if  $t > 0$  for  $t::real$**   
 $\langle proof \rangle$

### 5.1.5 Properties of $\$p-\{t\&x\}$

**lemma p-0-1:**  $\$p-\{0\&x\} = 1$   
 $\langle proof \rangle$

**lemma p-nonneg[simp]:**  $\$p-\{t\&x\} \geq 0$  **for  $t::real$**   
 $\langle proof \rangle$

**lemma p-le-1[simp]:**  $\$p-\{t\&x\} \leq 1$  **for  $t::real$**   
 $\langle proof \rangle$

**lemma p-0-equiv:**  $\$p-\{t\&x\} = 0 \longleftrightarrow x+t \geq \$\psi$  **for  $t::real$**   
 $\langle proof \rangle$

**lemma p-PTx:**  $\$p-\{t\&x\} = \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi > t \mid T x \xi > 0)$  **for  $t::real$**   
 $\langle proof \rangle$

**lemma p-PX:**  $\$p-\{t\&x\} = \mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x + t) / \mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x)$  **if  $t \geq 0$  for  $t::real$**   
 $\langle proof \rangle$

**lemma p-mult:**  $\$p-\{t+t' \& x\} = \$p-\{t\&x\} * \$p-\{t' \& x+t\}$   
**if  $t \geq 0 t' \geq 0 x+t < \$\psi$  for  $t t' :: real$**   
 $\langle proof \rangle$

**lemma p-PTx-ge-ccdf-isCont:**  $\$p-\{t\&x\} = \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \geq t \mid T x \xi > 0)$   
**if isCont (ccdf (distr \mathfrak{M} borel X)) (x+t)  $t > 0$  for  $t::real$**

*<proof>*

**end**

### 5.1.6 Properties of Survival Function for $X$

**lemma** *ccdfX-continuous-unborn[simp]: continuous-on  $\{..0\}$  (ccdf (distr  $\mathfrak{M}$  borel  $X$ ))*

*<proof>*

**lemma** *ccdfX-differentiable-unborn[simp]: (ccdf (distr  $\mathfrak{M}$  borel  $X$ )) differentiable-on  $\{..0\}$*

*<proof>*

**lemma** *ccdfX-has-real-derivative-0-unborn:*

*(ccdf (distr  $\mathfrak{M}$  borel  $X$ ) has-real-derivative 0) (at  $x$ ) if  $x < 0$  for  $x::real$*

*<proof>*

**lemma** *ccdfX-integrable-Icc:*

*set-integrable lborel  $\{a..b\}$  (ccdf (distr  $\mathfrak{M}$  borel  $X$ )) for  $a b :: real$*

*<proof>*

**corollary** *ccdfX-integrable-on-Icc:*

*ccdf (distr  $\mathfrak{M}$  borel  $X$ ) integrable-on  $\{a..b\}$  for  $a b :: real$*

*<proof>*

**lemma** *ccdfX-p: ccdf (distr  $\mathfrak{M}$  borel  $X$ )  $x = p - \{x \leq 0\}$  for  $x::real$*

*<proof>*

### 5.1.7 Introduction of Cumulative Distributive Function for $X$

**lemma** *cdfX-0-0: cdf (distr  $\mathfrak{M}$  borel  $X$ ) 0 = 0*

*<proof>*

**lemma** *cdfX-unborn-0: cdf (distr  $\mathfrak{M}$  borel  $X$ )  $x = 0$  if  $x \leq 0$*

*<proof>*

**lemma** *cdfX-beyond-1: cdf (distr  $\mathfrak{M}$  borel  $X$ )  $x = 1$  if  $x > \psi$  for  $x::real$*

*<proof>*

**lemma** *cdfX-psi-1: cdf (distr  $\mathfrak{M}$  borel  $X$ ) (real-of-ereal  $\psi$ ) = 1 if  $\psi < \infty$*

*<proof>*

**lemma** *cdfX-1-equiv: cdf (distr  $\mathfrak{M}$  borel  $X$ )  $x = 1 \iff x \geq \psi$  for  $x::real$*

*<proof>*

### 5.1.8 Properties of Cumulative Distributive Function for $T(x)$

**context**

**fixes**  $x::real$

**assumes**  $x\text{-lt-psi}$ [simp]:  $x < \psi$   
**begin**

**interpretation**  $\text{alive}\text{-PS}$ : prob-space  $\mathfrak{M} \mid \text{alive } x$   
 $\langle \text{proof} \rangle$

**interpretation**  $\text{distr}\text{Tx}\text{-RD}$ : real-distribution  $\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x) \langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-cond}\text{-prob}$ :

$\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)) t = \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \leq t \mid T x \xi > 0)$  **for**  
 $t :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-0-0}$ :  $\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)) 0 = 0$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-nonpos}\text{-0}$ :  $\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)) t = 0$  **if**  $t \leq 0$  **for**  
 $t :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-1-equiv}$ :  $\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)) t = 1 \iff x+t \geq \psi$   
**for**  $t :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-continuous-on-nonpos}$ [simp]:  
 $\text{continuous-on } \{..0\} (\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)))$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-differentiable-on-nonpos}$ [simp]:  
 $(\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x))) \text{ differentiable-on } \{..0\}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-has-real-derivative-0-at-neg}$ :  
 $(\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)) \text{ has-real-derivative } 0) (\text{at } t)$  **if**  $t < 0$  **for**  
 $t :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-integrable-Icc}$ :  
 $\text{set-integrable lborel } \{a..b\} (\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)))$  **for**  $a b :: \text{real}$   
 $\langle \text{proof} \rangle$

**corollary**  $\text{cdf}\text{Tx}\text{-integrable-on-Icc}$ :  
 $\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)) \text{ integrable-on } \{a..b\}$  **for**  $a b :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{cdf}\text{Tx}\text{-PX}$ :  
 $\text{cdf } (\text{distr } (\mathfrak{M} \mid \text{alive } x) \text{ borel } (T x)) t = \mathcal{P}(\xi \text{ in } \mathfrak{M}. x < X \xi \wedge X \xi \leq x+t) /$   
 $\mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x)$   
**for**  $t :: \text{real}$

*<proof>*

**lemma** *cdfT0-eq-cdfX*:  $\text{cdf} (\text{distr } (\mathfrak{M} \downarrow \text{alive } 0) \text{ borel } (T\ 0)) = \text{cdf} (\text{distr } \mathfrak{M} \text{ borel } X)$

*<proof>*

**lemma** *continuous-cdfX-cdfTx*:

*continuous (at (x+t) within {x..}) (cdf (distr  $\mathfrak{M}$  borel X))  $\longleftrightarrow$*

*continuous (at t within {0..}) (cdf (distr ( $\mathfrak{M} \downarrow \text{alive } x$ ) borel (T x)))*

**if**  $t \geq 0$  **for**  $t::\text{real}$

*<proof>*

**lemma** *isCont-cdfX-cdfTx*:

*isCont (cdf (distr  $\mathfrak{M}$  borel X)) (x+t)  $\longleftrightarrow$*

*isCont (cdf (distr ( $\mathfrak{M} \downarrow \text{alive } x$ ) borel (T x))) t*

**if**  $t > 0$  **for**  $t::\text{real}$

*<proof>*

**lemma** *has-real-derivative-cdfX-cdfTx*:

*((cdf (distr  $\mathfrak{M}$  borel X)) has-real-derivative D) (at (x+t))  $\longleftrightarrow$*

*((cdf (distr ( $\mathfrak{M} \downarrow \text{alive } x$ ) borel (T x))) has-real-derivative (D /  $\mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x)$ )) (at t)*

**if**  $t > 0$  **for**  $t D :: \text{real}$

*<proof>*

**lemma** *differentiable-cdfX-cdfTx*:

*(cdf (distr  $\mathfrak{M}$  borel X)) differentiable at (x+t)  $\longleftrightarrow$*

*(cdf (distr ( $\mathfrak{M} \downarrow \text{alive } x$ ) borel (T x))) differentiable at t*

**if**  $t > 0$  **for**  $t::\text{real}$

*<proof>*

### 5.1.9 Properties of $\$q\{-t\&x\}$

**lemma** *q-nonpos-0*:  $\$q\{-t\&x\} = 0$  **if**  $t \leq 0$  **for**  $t::\text{real}$

*<proof>*

**corollary** *q-0-0*:  $\$q\{0\&x\} = 0$

*<proof>*

**lemma** *q-nonneg[simp]*:  $\$q\{-t\&x\} \geq 0$  **for**  $t::\text{real}$

*<proof>*

**lemma** *q-le-1[simp]*:  $\$q\{-t\&x\} \leq 1$  **for**  $t::\text{real}$

*<proof>*

**lemma** *q-1-equiv*:  $\$q\{-t\&x\} = 1 \longleftrightarrow x+t \geq \$\psi$  **for**  $t::\text{real}$

*<proof>*

**lemma** *q-PTx*:  $\$q\{-t\&x\} = \mathcal{P}(\xi \text{ in } \mathfrak{M}. T\ x\ \xi \leq t \mid T\ x\ \xi > 0)$  **for**  $t::\text{real}$



*<proof>*

**lemma** *q-PX*:  $\$q\{-t\&x\} = \mathcal{P}(\xi \text{ in } \mathfrak{M}. x < X \xi \wedge X \xi \leq x + t) / \mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x)$

*<proof>*

**lemma** *q-defer-0-q[simp]*:  $\$q\{0|t\&x\} = \$q\{-t\&x\}$  **for**  $t::\text{real}$

*<proof>*

**lemma** *q-defer-0-0*:  $\$q\{f|0\&x\} = 0$  **for**  $f::\text{real}$

*<proof>*

**lemma** *q-defer-nonneg[simp]*:  $\$q\{f|t\&x\} \geq 0$  **for**  $f t :: \text{real}$

*<proof>*

**lemma** *q-defer-q*:  $\$q\{f|t\&x\} = \$q\{f+t \& x\} - \$q\{f\&x\}$  **if**  $t \geq 0$  **for**  $f t :: \text{real}$

*<proof>*

**corollary** *q-defer-le-1[simp]*:  $\$q\{f|t\&x\} \leq 1$  **if**  $t \geq 0$  **for**  $f t :: \text{real}$

*<proof>*

**lemma** *q-defer-PTx*:  $\$q\{f|t\&x\} = \mathcal{P}(\xi \text{ in } \mathfrak{M}. f < T x \xi \wedge T x \xi \leq f + t \mid T x \xi > 0)$

**if**  $t \geq 0$  **for**  $f t :: \text{real}$

*<proof>*

**lemma** *q-defer-PX*:  $\$q\{f|t\&x\} = \mathcal{P}(\xi \text{ in } \mathfrak{M}. x + f < X \xi \wedge X \xi \leq x + f + t) / \mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi > x)$

**if**  $f \geq 0 t \geq 0$  **for**  $f t :: \text{real}$

*<proof>*

**lemma** *q-defer-old-0*:  $\$q\{f|t\&x\} = 0$  **if**  $x+f \geq \psi t \geq 0$  **for**  $f t :: \text{real}$

*<proof>*

**end**

### 5.1.10 Properties of Cumulative Distributive Function for $X$

**lemma** *cdfX-continuous-unborn[simp]*: *continuous-on*  $\{..0\}$  (*cdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ))

*<proof>*

**lemma** *cdfX-differentiable-unborn[simp]*: (*cdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ )) *differentiable-on*  $\{..0\}$

*<proof>*

**lemma** *cdfX-has-real-derivative-0-unborn*:

(*cdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ) *has-real-derivative* 0) (*at*  $x$ ) **if**  $x < 0$  **for**  $x::\text{real}$

*<proof>*

**lemma** *cdfX-integrable-Icc*:  
*set-integrable lborel {a..b} (cdf (distr  $\mathfrak{M}$  borel X))* **for**  $a\ b :: \text{real}$   
 ⟨*proof*⟩

**corollary** *cdfX-integrable-on-Icc*:  
*cdf (distr  $\mathfrak{M}$  borel X) integrable-on {a..b}* **for**  $a\ b :: \text{real}$   
 ⟨*proof*⟩

**lemma** *cdfX-q*: *cdf (distr  $\mathfrak{M}$  borel X) x =  $\$q\{x\&0\}$*  **if**  $x \geq 0$  **for**  $x :: \text{real}$   
 ⟨*proof*⟩

### 5.1.11 Relations between $\$p\{t\&x\}$ and $\$q\{t\&x\}$

**context**  
**fixes**  $x :: \text{real}$   
**assumes**  $x\text{-lt-}\psi[\text{simp}]$ :  $x < \psi$   
**begin**

**interpretation** *alivex-PS*: *prob-space  $\mathfrak{M} \mid \text{alive } x$*   
 ⟨*proof*⟩

**interpretation** *distrTx-RD*: *real-distribution distr ( $\mathfrak{M} \mid \text{alive } x$ ) borel (Tx)* ⟨*proof*⟩

**lemma** *p-q-1*:  $\$p\{t\&x\} + \$q\{t\&x\} = 1$  **for**  $t :: \text{real}$   
 ⟨*proof*⟩

**lemma** *q-defer-p*:  $\$q\{f|t\&x\} = \$p\{f\&x\} - \$p\{f+t\&x\}$  **if**  $t \geq 0$  **for**  $f\ t :: \text{real}$   
 ⟨*proof*⟩

**lemma** *q-defer-p-q-defer*:  $\$p\{f\&x\} * \$q\{f'|t\&x+f\} = \$q\{f+f'|t\&x\}$   
**if**  $x+f < \psi$   $f \geq 0$   $f' \geq 0$   $t \geq 0$  **for**  $f\ f'\ t :: \text{real}$   
 ⟨*proof*⟩

**lemma** *q-defer-pq*:  $\$q\{f|t\&x\} = \$p\{f\&x\} * \$q\{t\&x+f\}$   
**if**  $x+f < \psi$   $t \geq 0$   $f \geq 0$  **for**  $f\ t :: \text{real}$   
 ⟨*proof*⟩

### 5.1.12 Properties of Life Expectation

**lemma** *e-nonneg*:  $\$e'\circ\text{-}x \geq 0$   
 ⟨*proof*⟩

**lemma** *e-P*:  $\$e'\circ\text{-}x =$   
*MM-PS.expectation* ( $\lambda\xi$ . *indicator (alive x)  $\xi * Tx\ \xi$* ) /  $\mathcal{P}(\xi \text{ in } \mathfrak{M}. Tx\ \xi > 0)$   
 ⟨*proof*⟩

**proposition** *nn-integral-T-p*:  
 $(\int^{+\xi}. \text{ennreal } (Tx\ \xi) \partial(\mathfrak{M} \mid \text{alive } x)) = (\int^{+t \in \{0..\}}. \text{ennreal } (\$p\{t\&x\}) \partial\text{lborel})$   
 ⟨*proof*⟩

**lemma** *nn-integral-T-pos*:  $(\int^{+\xi}. \text{ennreal } (T x \xi) \partial(\mathfrak{M} \downarrow \text{alive } x)) > 0$   
 ⟨proof⟩

**lemma** *e-pos-Tx*:  $\$e^{\circ-x} > 0$  if integrable  $(\mathfrak{M} \downarrow \text{alive } x) (T x)$   
 ⟨proof⟩

**proposition** *e-LBINT-p*:  $\$e^{\circ-x} = (\text{LBINT } t:\{0..n\}. \$p-\{t\&x\})$   
 — Note that  $0 = 0$  holds when the integral diverges.  
 ⟨proof⟩

**corollary** *e-integral-p*:  $\$e^{\circ-x} = \text{integral } \{0..n\} (\lambda t. \$p-\{t\&x\})$   
 — Note that  $0 = 0$  holds when the integral diverges.  
 ⟨proof⟩

**lemma** *e-pos*:  $\$e^{\circ-x} > 0$  if set-integrable lborel  $\{0..n\} (\lambda t. \$p-\{t\&x\})$   
 ⟨proof⟩

**corollary** *e-pos'*:  $\$e^{\circ-x} > 0$  if  $(\lambda t. \$p-\{t\&x\})$  integrable-on  $\{0..n\}$   
 ⟨proof⟩

**lemma** *e-LBINT-p-Icc*:  $\$e^{\circ-x} = (\text{LBINT } t:\{0..n\}. \$p-\{t\&x\})$  if  $x+n \geq \$\psi$  for  $n::\text{real}$   
 ⟨proof⟩

**lemma** *e-integral-p-Icc*:  $\$e^{\circ-x} = \text{integral } \{0..n\} (\lambda t. \$p-\{t\&x\})$  if  $x+n \geq \$\psi$  for  $n::\text{real}$   
 ⟨proof⟩

**lemma** *temp-e-le-n*:  $\$e^{\circ-\{x:n\}} \leq n$  if  $n \geq 0$  for  $n::\text{real}$   
 ⟨proof⟩

**lemma** *temp-e-P*:  $\$e^{\circ-\{x:n\}} =$   
 $\text{MM-PS.expectation } (\lambda\xi. \text{indicator } (\text{alive } x) \xi * \min (T x \xi) n) / \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x$   
 $\xi > 0)$   
 if  $n \geq 0$  for  $n::\text{real}$   
 ⟨proof⟩

**lemma** *temp-e-LBINT-p*:  $\$e^{\circ-\{x:n\}} = (\text{LBINT } t:\{0..n\}. \$p-\{t\&x\})$  if  $n \geq 0$  for  $n::\text{real}$   
 ⟨proof⟩

**lemma** *temp-e-integral-p*:  $\$e^{\circ-\{x:n\}} = \text{integral } \{0..n\} (\lambda t. \$p-\{t\&x\})$  if  $n \geq 0$  for  $n::\text{real}$   
 ⟨proof⟩

**lemma** *e-eq-temp*:  $\$e^{\circ-x} = \$e^{\circ-\{x:n\}}$  if  $n \geq 0$   $x+n \geq \$\psi$  for  $n::\text{real}$   
 ⟨proof⟩

**lemma** *curt-e-P*:  $\$e-x =$

*MM-PS.expectation*  $(\lambda\xi. \text{indicator } (\text{alive } x) \xi * \lfloor T x \xi \rfloor) / \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi > 0)$

*<proof>*

**lemma** *curt-e-sum-P*:  $\$e-x = (\sum k. \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \geq k + 1 \mid T x \xi > 0))$

**if** *summable*  $(\lambda k. \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \geq k + 1 \mid T x \xi > 0))$

*<proof>*

**lemma** *curt-e-sum-P-finite*:  $\$e-x = (\sum k < n. \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \geq k + 1 \mid T x \xi > 0))$

**if**  $x+n+1 > \$\psi$  **for**  $n::\text{nat}$

*<proof>*

**lemma** *curt-e-sum-p*:  $\$e-x = (\sum k. \$p-\{k+1\&x\})$

**if** *summable*  $(\lambda k. \$p-\{k+1\&x\}) \wedge k::\text{nat}. \text{isCont } (\lambda t. \$p-\{t\&x\}) (k+1)$

*<proof>*

**lemma** *curt-e-rec*:  $\$e-x = \$p-x * (1 + \$e-(x+1))$

**if** *summable*  $(\lambda k. \$p-\{k+1\&x\}) \wedge k::\text{nat}. \text{isCont } (\lambda t. \$p-\{t\&x\}) (\text{real } k + 1) x+1 < \$\psi$

*<proof>*

**lemma** *curt-e-sum-p-finite*:  $\$e-x = (\sum k < n. \$p-\{k+1\&x\})$

**if**  $\wedge k::\text{nat}. k < n \implies \text{isCont } (\lambda t. \$p-\{t\&x\}) (\text{real } k + 1) x+n+1 > \$\psi$  **for**  $n::\text{nat}$

*<proof>*

**lemma** *temp-curt-e-P*:  $\$e-\{x:n\} =$

*MM-PS.expectation*  $(\lambda\xi. \text{indicator } (\text{alive } x) \xi * \lfloor \min (T x \xi) n \rfloor) / \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi > 0)$

**if**  $n \geq 0$  **for**  $n::\text{real}$

*<proof>*

**lemma** *temp-curt-e-sum-P*:  $\$e-\{x:n\} = (\sum k < n. \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \geq k + 1 \mid T x \xi > 0))$  **for**  $n::\text{nat}$

*<proof>*

**corollary** *curt-e-eq-temp*:  $\$e-x = \$e-\{x:n\}$  **if**  $x+n+1 > \$\psi$  **for**  $n::\text{nat}$

*<proof>*

**lemma** *temp-curt-e-sum-p*:  $\$e-\{x:n\} = (\sum k < n. \$p-\{k+1\&x\})$

**if**  $\wedge k::\text{nat}. k < n \implies \text{isCont } (\lambda t. \$p-\{t\&x\}) (\text{real } k + 1)$  **for**  $n::\text{nat}$

*<proof>*

**lemma** *temp-curt-e-rec*:  $\$e-\{x:n\} = \$p-x * (1 + \$e-\{x+1:n-1\})$

**if**  $\wedge k::\text{nat}. k < n \implies \text{isCont } (\lambda t. \$p-\{t\&x\}) (\text{real } k + 1) x+1 < \$\psi$   $n \neq 0$  **for**  $n::\text{nat}$

*<proof>*

**end**

**lemma** *p-set-integrable-shift*:

*set-integrable lborel {0..} (λt. \$p-{t&x})*  $\longleftrightarrow$  *set-integrable lborel {0..} (λt. \$p-{t&x})*  
**if**  $x < \psi$  **for**  $x :: \text{real}$   
*<proof>*

**lemma** *e-p-e*:  $\$e^{\circ-x} = \$e^{\circ-\{x:n\}} + \$p-\{n&x\} * \$e^{\circ-(x+n)}$

**if** *set-integrable lborel {0..} (λt. \$p-{t&x})*  $n \geq 0$   $x+n < \psi$  **for**  $x n :: \text{real}$   
*<proof>*

**proposition** *x-ex-mono*:  $x + \$e^{\circ-x} \leq y + \$e^{\circ-y}$  **if**  $x \leq y$   $y < \psi$  **for**  $x y :: \text{real}$   
*<proof>*

**proposition** *x-ex-const-equiv*:  $x + \$e^{\circ-x} = y + \$e^{\circ-y} \longleftrightarrow \$q-\{y-x&x\} = 0$

**if** *set-integrable lborel {0..} (λt. \$p-{t&0})*  $x \leq y$   $y < \psi$  **for**  $x y :: \text{real}$   
*<proof>*

**end**

## 5.2 Piecewise Differentiable Survival Function

**locale** *smooth-survival-function = survival-model +*

**assumes** *ccdfX-piecewise-differentiable[simp]*:

*(ccdf (distr  $\mathfrak{M}$  borel X)) piecewise-differentiable-on UNIV*

**begin**

**interpretation** *distrX-RD*: *real-distribution distr  $\mathfrak{M}$  borel X*

*<proof>*

### 5.2.1 Properties of Survival Function for X

**lemma** *ccdfX-continuous[simp]*: *continuous-on UNIV (ccdf (distr  $\mathfrak{M}$  borel X))*

*<proof>*

**corollary** *ccdfX-borel-measurable[measurable]*: *ccdf (distr  $\mathfrak{M}$  borel X)  $\in$  borel-measurable borel*

*<proof>*

**lemma** *ccdfX-nondifferentiable-finite-set[simp]*:

*finite {x.  $\neg$  ccdf (distr  $\mathfrak{M}$  borel X) differentiable at x}*

*<proof>*

**lemma** *ccdfX-differentiable-open-set*: *open {x. ccdf (distr  $\mathfrak{M}$  borel X) differentiable at x}*

*<proof>*

**lemma** *ccdfX-differentiable-borel-set[measurable, simp]*:

*{x. ccdf (distr  $\mathfrak{M}$  borel X) differentiable at x}  $\in$  sets borel*

*<proof>*

**lemma** *ccdfX-differentiable-AE*:

*AE x in lborel. (ccdf (distr  $\mathfrak{M}$  borel X)) differentiable at x*

*<proof>*

**lemma** *deriv-ccdfX-measurable[measurable]*: *deriv (ccdf (distr  $\mathfrak{M}$  borel X))  $\in$  borel-measurable borel*

*<proof>*

### 5.2.2 Properties of Cumulative Distributive Function for $X$

**lemma** *cdfX-piecewise-differentiable[simp]*:

*(cdf (distr  $\mathfrak{M}$  borel X)) piecewise-differentiable-on UNIV*

*<proof>*

**lemma** *cdfX-continuous[simp]*: *continuous-on UNIV (cdf (distr  $\mathfrak{M}$  borel X))*

*<proof>*

**corollary** *cdfX-borel-measurable[measurable]*: *cdf (distr  $\mathfrak{M}$  borel X)  $\in$  borel-measurable borel*

*<proof>*

**lemma** *cdfX-nondifferentiable-finite-set[simp]*:

*finite {x.  $\neg$  cdf (distr  $\mathfrak{M}$  borel X) differentiable at x}*

*<proof>*

**lemma** *cdfX-differentiable-open-set*: *open {x. cdf (distr  $\mathfrak{M}$  borel X) differentiable at x}*

*<proof>*

**lemma** *cdfX-differentiable-borel-set[measurable, simp]*:

*{x. cdf (distr  $\mathfrak{M}$  borel X) differentiable at x}  $\in$  sets borel*

*<proof>*

**lemma** *cdfX-differentiable-AE*:

*AE x in lborel. (cdf (distr  $\mathfrak{M}$  borel X)) differentiable at x*

*<proof>*

**lemma** *deriv-cdfX-measurable[measurable]*: *deriv (cdf (distr  $\mathfrak{M}$  borel X))  $\in$  borel-measurable borel*

*<proof>*

### 5.2.3 Introduction of Probability Density Functions of $X$ and

$T(x)$

**definition** *pdfX :: real  $\Rightarrow$  real*

**where** *pdfX x  $\equiv$  if cdf (distr  $\mathfrak{M}$  borel X) differentiable at x*

*then deriv (cdf (distr  $\mathfrak{M}$  borel X)) x else 0*

— This function is defined to be always nonnegative for future application.

**definition**  $pdfT :: real \Rightarrow real \Rightarrow real$

**where**  $pdfT\ x\ t \equiv$  if  $cdf\ (distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x))$  differentiable at  $t$   
then  $deriv\ (cdf\ (distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x)))\ t$  else  $0$

— This function is defined to be always nonnegative for future application.

**lemma**  $pdfX$ -measurable[measurable]:  $pdfX \in borel$ -measurable borel  
(proof)

**lemma** distributed- $pdfX$ : distributed  $\mathfrak{M}$  lborel  $X$   $pdfX$   
(proof)

**lemma**  $pdfT0$ - $X$ :  $pdfT\ 0 = pdfX$   
(proof)

#### 5.2.4 Properties of Survival Function for $T(x)$

**context**

**fixes**  $x::real$

**assumes**  $x$ -lt-psi[simp]:  $x < \psi$

**begin**

**interpretation**  $aliveX$ -PS: prob-space  $\mathfrak{M} \mid alive\ x$   
(proof)

**interpretation**  $distrTx$ -RD: real-distribution  $distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x)$  (proof)

**lemma**  $ccdfTx$ -continuous-on-nonneg[simp]:  
continuous-on  $\{0..\}$  ( $ccdf\ (distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x))$ )  
(proof)

**lemma**  $ccdfTx$ -continuous[simp]: continuous-on UNIV ( $ccdf\ (distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x))$ )  
(proof)

**corollary**  $ccdfTx$ -borel-measurable[measurable]:  
 $ccdf\ (distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x)) \in borel$ -measurable borel  
(proof)

**lemma**  $ccdfTx$ -nondifferentiable-finite-set[simp]:  
finite  $\{t. \neg cdf\ (distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x))\}$  differentiable at  $t$   
(proof)

**lemma**  $ccdfTx$ -differentiable-open-set:  
open  $\{t. cdf\ (distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x))\}$  differentiable at  $t$   
(proof)

**lemma**  $ccdfTx$ -differentiable-borel-set[measurable, simp]:  
 $\{t. cdf\ (distr\ (\mathfrak{M} \mid alive\ x)\ borel\ (T\ x))\}$  differentiable at  $t \in sets\ borel$

*<proof>*

**lemma** *ccdfTx-differentiable-AE*:

*AE t in lborel. (ccdf (distr (M | alive x) borel (T x))) differentiable at t*  
*<proof>*

**lemma** *ccdfTx-piecewise-differentiable[simp]*:

*(ccdf (distr (M | alive x) borel (T x))) piecewise-differentiable-on UNIV*  
*<proof>*

**lemma** *deriv-ccdfTx-measurable[measurable]*:

*deriv (ccdf (distr (M | alive x) borel (T x))) ∈ borel-measurable borel*  
*<proof>*

### 5.2.5 Properties of Cumulative Distributive Function for $T(x)$

**lemma** *cdfTx-continuous[simp]*:

*continuous-on UNIV (cdf (distr (M | alive x) borel (T x)))*  
*<proof>*

**corollary** *cdfTx-borel-measurable[measurable]*:

*cdf (distr (M | alive x) borel (T x)) ∈ borel-measurable borel*  
*<proof>*

**lemma** *cdfTx-nondifferentiable-finite-set[simp]*:

*finite {t. ¬ cdf (distr (M | alive x) borel (T x)) differentiable at t}*  
*<proof>*

**lemma** *cdfTx-differentiable-open-set*:

*open {t. cdf (distr (M | alive x) borel (T x)) differentiable at t}*  
*<proof>*

**lemma** *cdfTx-differentiable-borel-set[measurable, simp]*:

*{t. cdf (distr (M | alive x) borel (T x)) differentiable at t} ∈ sets borel*  
*<proof>*

**lemma** *cdfTx-differentiable-AE*:

*AE t in lborel. (cdf (distr (M | alive x) borel (T x))) differentiable at t*  
*<proof>*

**lemma** *cdfTx-piecewise-differentiable[simp]*:

*(cdf (distr (M | alive x) borel (T x))) piecewise-differentiable-on UNIV*  
*<proof>*

**lemma** *deriv-cdfTx-measurable[measurable]*:

*deriv (cdf (distr (M | alive x) borel (T x))) ∈ borel-measurable borel*  
*<proof>*



## 5.2.6 Properties of Probability Density Function of $T(x)$

**lemma** *pdfTx-nonneg*:  $\text{pdf}T\ x\ t \geq 0$  for  $t::\text{real}$

*<proof>*

**lemma** *pdfTx-neg-0*:  $\text{pdf}T\ x\ t = 0$  if  $t < 0$  for  $t::\text{real}$

*<proof>*

**lemma** *pdfTx-0-0*:  $\text{pdf}T\ x\ 0 = 0$

*<proof>*

**lemma** *pdfTx-nonpos-0*:  $\text{pdf}T\ x\ t = 0$  if  $t \leq 0$  for  $t::\text{real}$

*<proof>*

**lemma** *pdfTx-beyond-0*:  $\text{pdf}T\ x\ t = 0$  if  $x+t \geq \psi$  for  $t::\text{real}$

*<proof>*

**lemma** *pdfTx-pdfX*:  $\text{pdf}T\ x\ t = \text{pdf}X\ (x+t) / \mathcal{P}(\xi \text{ in } \mathfrak{M}. X\ \xi > x)$  if  $t > 0$  for

$t::\text{real}$

*<proof>*

**lemma** *pdfTx-measurable[measurable]*:  $\text{pdf}T\ x \in \text{borel-measurable borel}$

*<proof>*

**lemma** *distributed-pdfTx*:  $\text{distributed } (\mathfrak{M} \mid \text{alive } x) \text{ lborel } (T\ x) (\text{pdf}T\ x)$

*<proof>*

**lemma** *nn-integral-pdfTx-1*:  $(\int^+ s. \text{pdf}T\ x\ s\ \partial\text{lborel}) = 1$

*<proof>*

**corollary** *has-bochner-integral-pdfTx-1*:  $\text{has-bochner-integral lborel } (\text{pdf}T\ x)\ 1$

*<proof>*

**corollary** *LBINT-pdfTx-1*:  $(\text{LBINT } s. \text{pdf}T\ x\ s) = 1$

*<proof>*

**corollary** *pdfTx-has-integral-1*:  $(\text{pdf}T\ x\ \text{has-integral } 1) \text{ UNIV}$

*<proof>*

**lemma** *set-nn-integral-pdfTx-1*:  $(\int^+ s \in \{0..\}. \text{pdf}T\ x\ s\ \partial\text{lborel}) = 1$

*<proof>*

**corollary** *has-bochner-integral-pdfTx-1-nonpos*:

$\text{has-bochner-integral lborel } (\lambda s. \text{pdf}T\ x\ s * \text{indicator } \{0..\} s)\ 1$

*<proof>*

**corollary** *set-LBINT-pdfTx-1*:  $(\text{LBINT } s:\{0..\}. \text{pdf}T\ x\ s) = 1$

*<proof>*

**corollary** *pdfTx-has-integral-1-nonpos*:  $(\text{pdf}T\ x\ \text{has-integral } 1) \{0..\}$

*<proof>*

**lemma** *set-nn-integral-pdfTx-PTx*:  $(\int^{+s \in A} \text{pdfT } x \text{ } s \text{ } \partial \text{lborel}) = \mathcal{P}(\xi \text{ in } \mathfrak{M}. T \text{ } x \text{ } \xi \in A \mid T \text{ } x \text{ } \xi > 0)$

**if**  $A \in \text{sets lborel}$  **for**  $A :: \text{real set}$

*<proof>*

**lemma** *pdfTx-set-integrable*: *set-integrable lborel A (pdfT x)* **if**  $A \in \text{sets lborel}$

*<proof>*

**lemma** *set-integral-pdfTx-PTx*:  $(\text{LBINT } s:A. \text{pdfT } x \text{ } s) = \mathcal{P}(\xi \text{ in } \mathfrak{M}. T \text{ } x \text{ } \xi \in A \mid T \text{ } x \text{ } \xi > 0)$

**if**  $A \in \text{sets lborel}$  **for**  $A :: \text{real set}$

*<proof>*

**lemma** *pdfTx-has-integral-PTx*:  $(\text{pdfT } x \text{ has-integral } \mathcal{P}(\xi \text{ in } \mathfrak{M}. T \text{ } x \text{ } \xi \in A \mid T \text{ } x \text{ } \xi > 0)) \text{ } A$

**if**  $A \in \text{sets lborel}$  **for**  $A :: \text{real set}$

*<proof>*

**corollary** *pdfTx-has-integral-PTx-Icc*:

$(\text{pdfT } x \text{ has-integral } \mathcal{P}(\xi \text{ in } \mathfrak{M}. a \leq T \text{ } x \text{ } \xi \wedge T \text{ } x \text{ } \xi \leq b \mid T \text{ } x \text{ } \xi > 0)) \{a..b\}$  **for**  
 $a \text{ } b :: \text{real}$

*<proof>*

**corollary** *pdfTx-integrable-on-Icc*: *pdfT x integrable-on {a..b}* **for**  $a \text{ } b :: \text{real}$

*<proof>*

**end**

## 5.2.7 Properties of Probability Density Function of $X$

**lemma** *pdfX-nonneg*: *pdfX x ≥ 0* **for**  $x :: \text{real}$

*<proof>*

**lemma** *pdfX-nonpos-0*: *pdfX x = 0* **if**  $x \leq 0$  **for**  $x :: \text{real}$

*<proof>*

**lemma** *pdfX-beyond-0*: *pdfX x = 0* **if**  $x \geq \psi$  **for**  $x :: \text{real}$

*<proof>*

**lemma** *nn-integral-pdfX-1*:  $\text{integral}^N \text{ lborel } \text{pdfX} = 1$

*<proof>*

**corollary** *has-bochner-integral-pdfX-1*: *has-bochner-integral lborel pdfX 1*

*<proof>*

**corollary** *LBINT-pdfX-1*:  $(\text{LBINT } s. \text{pdfX } s) = 1$

*<proof>*

**corollary** *pdfX-has-integral-1*: (*pdfX has-integral 1*) UNIV  
 ⟨*proof*⟩

**lemma** *set-nn-integral-pdfX-PX*: *set-nn-integral lborel A pdfX = P(ξ in M. X ξ ∈ A)*  
 if *A ∈ sets lborel for A :: real set*  
 ⟨*proof*⟩

**lemma** *pdfX-set-integrable*: *set-integrable lborel A pdfX if A ∈ sets lborel for A :: real set*  
 ⟨*proof*⟩

**lemma** *set-integral-pdfX-PX*: (*LBINT s:A. pdfX s = P(ξ in M. X ξ ∈ A)*)  
 if *A ∈ sets lborel for A :: real set*  
 ⟨*proof*⟩

**lemma** *pdfX-has-integral-PX*: (*pdfX has-integral P(ξ in M. X ξ ∈ A)*) *A*  
 if *A ∈ sets lborel for A :: real set*  
 ⟨*proof*⟩

**corollary** *pdfX-has-integral-PX-Icc*: (*pdfX has-integral P(ξ in M. a ≤ X ξ ∧ X ξ ≤ b)*) {*a..b*}  
 for *a b :: real*  
 ⟨*proof*⟩

**corollary** *pdfX-integrable-on-Icc*: *pdfX integrable-on {a..b} for a b :: real*  
 ⟨*proof*⟩

## 5.2.8 Relations between Life Expectation and Probability Density Function

**context**  
 fixes *x::real*  
 assumes *x-lt-psi[simp]*: *x < ψ*  
**begin**

**interpretation** *alivex-PS*: *prob-space M | alive x*  
 ⟨*proof*⟩

**interpretation** *distrTx-RD*: *real-distribution distr (M | alive x) borel (T x)* ⟨*proof*⟩

**proposition** *nn-integral-T-pdfT*:  
 ( $\int^{+\xi}. \text{ennreal } (g (T x \xi)) \partial(\mathfrak{M} \upharpoonright \text{alive } x) = (\int^{+s \in \{0.. \}}. \text{ennreal } (\text{pdfT } x s * g s) \partial \text{lborel})$ )  
 if *g ∈ borel-measurable lborel for g :: real ⇒ real*  
 ⟨*proof*⟩

**lemma** *expectation-LBINT-pdfT-nonneg*:

*alive-PS.expectation*  $(\lambda\xi. g (T x \xi)) = (LBINT s:\{0..\}. pdfT x s * g s)$   
**if**  $\bigwedge s. s \geq 0 \implies g s \geq 0$   $g \in \text{borel-measurable lborel}$  **for**  $g :: \text{real} \Rightarrow \text{real}$   
— Note that  $0 = 0$  holds when the integral diverges.  
⟨*proof*⟩

**corollary** *expectation-integral-pdfT-nonneg:*

*alive-PS.expectation*  $(\lambda\xi. g (T x \xi)) = \text{integral } \{0..\} (\lambda s. pdfT x s * g s)$   
**if**  $\bigwedge s. s \geq 0 \implies g s \geq 0$   $g \in \text{borel-measurable lborel}$  **for**  $g :: \text{real} \Rightarrow \text{real}$   
— Note that  $0 = 0$  holds when the integral diverges.  
⟨*proof*⟩

**proposition** *expectation-LBINT-pdfT:*

*alive-PS.expectation*  $(\lambda\xi. g (T x \xi)) = (LBINT s:\{0..\}. pdfT x s * g s)$   
**if** *set-integrable lborel*  $\{0..\} (\lambda s. pdfT x s * g s)$   $g \in \text{borel-measurable lborel}$   
**for**  $g :: \text{real} \Rightarrow \text{real}$   
⟨*proof*⟩

**corollary** *expectation-integral-pdfT:*

*alive-PS.expectation*  $(\lambda\xi. g (T x \xi)) = \text{integral } \{0..\} (\lambda s. pdfT x s * g s)$   
**if**  $(\lambda s. pdfT x s * g s)$  *absolutely-integrable-on*  $\{0..\}$   $g \in \text{borel-measurable lborel}$   
**for**  $g :: \text{real} \Rightarrow \text{real}$   
⟨*proof*⟩

**corollary** *e-LBINT-pdfT:*  $\$e^{\circ-x} = (LBINT s:\{0..\}. pdfT x s * s)$

— Note that  $0 = 0$  holds when the life expectation diverges.  
⟨*proof*⟩

**corollary** *e-integral-pdfT:*  $\$e^{\circ-x} = \text{integral } \{0..\} (\lambda s. pdfT x s * s)$

— Note that  $0 = 0$  holds when the life expectation diverges.  
⟨*proof*⟩

**end**

**corollary** *e-LBINT-pdfX:*  $\$e^{\circ-0} = (LBINT x:\{0..\}. pdfX x * x)$

— Note that  $0 = 0$  holds when the life expectation diverges.  
⟨*proof*⟩

**corollary** *e-integral-pdfX:*  $\$e^{\circ-0} = \text{integral } \{0..\} (\lambda x. pdfX x * x)$

— Note that  $0 = 0$  holds when the life expectation diverges.  
⟨*proof*⟩

### 5.2.9 Introduction of Force of Mortality

**definition** *force-mortal*  $:: \text{real} \Rightarrow \text{real}$  ( $\langle \$\mu' \rightarrow [101] 200$ )

**where**  $\$\mu-x \equiv \text{MM-PS.hazard-rate } X x$

**lemma** *mu-pdfX:*  $\$\mu-x = pdfX x / cdf$  (*distr*  $\mathfrak{M}$  *borel*  $X$ )  $x$

**if** (*cdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ )) *differentiable at*  $x$  **for**  $x :: \text{real}$   
⟨*proof*⟩

**lemma** *mu-unborn-0*:  $\mu-x = 0$  if  $x < 0$  for  $x::real$

*<proof>*

**lemma** *mu-beyond-0*:  $\mu-x = 0$  if  $x \geq \psi$  for  $x::real$

— Note that division by 0 is defined as 0 in Isabelle/HOL.

*<proof>*

**lemma** *mu-nonneg-differentiable*:  $\mu-x \geq 0$

if  $(cdf (distr \mathfrak{M} borel X))$  differentiable at  $x$  for  $x::real$

*<proof>*

**lemma** *mu-nonneg-AE*: AE  $x$  in  $lborel$ .  $\mu-x \geq 0$

*<proof>*

**lemma** *mu-measurable[measurable]*:  $(\lambda x. \mu-x) \in borel\text{-measurable } borel$

*<proof>*

**lemma** *mu-deriv-ccdf*:  $\mu-x = - deriv (ccdf (distr \mathfrak{M} borel X)) x / cdf (distr \mathfrak{M} borel X) x$

if  $(ccdf (distr \mathfrak{M} borel X))$  differentiable at  $x$   $x < \psi$  for  $x::real$

*<proof>*

**lemma** *mu-deriv-ln*:  $\mu-x = - deriv (\lambda x. \ln (ccdf (distr \mathfrak{M} borel X) x)) x$

if  $(ccdf (distr \mathfrak{M} borel X))$  differentiable at  $x$   $x < \psi$  for  $x::real$

*<proof>*

**lemma** *p-exp-integral-mu*:  $p-\{t\&x\} = exp (- integral \{x..x+t\} (\lambda y. \mu-y))$

if  $x \geq 0$   $t \geq 0$   $x+t < \psi$  for  $x t :: real$

*<proof>*

**corollary** *ccdfX-exp-integral-mu*:  $ccdf (distr \mathfrak{M} borel X) x = exp (- integral \{0..x\} (\lambda y. \mu-y))$

if  $0 \leq x \wedge x < \psi$  for  $x::real$

*<proof>*

## 5.2.10 Properties of Force of Mortality

**context**

fixes  $x::real$

assumes  $x\text{-lt-}\psi$ [*simp*]:  $x < \psi$

**begin**

**interpretation** *alivex-PS*:  $prob\text{-space } \mathfrak{M} \mid alive\ x$

*<proof>*

**interpretation** *distrTx-RD*:  $real\text{-distribution } distr (\mathfrak{M} \mid alive\ x) borel (T\ x)$  *<proof>*

**lemma** *hazard-rate-Tx-mu*:  $alivex\text{-PS.hazard-rate } (T\ x) t = \mu-(x+t)$

**if**  $t \geq 0$   $x+t < \psi$  **for**  $t::\text{real}$   
 ⟨proof⟩

**lemma** *pdfTx-p-mu*:  $\text{pdfT } x \ t = \mathbb{P}\{t \& x\} * \mathbb{M}\text{-}(x+t)$   
**if** (*cdf* (*distr* ( $\mathfrak{M} \downarrow \text{alive } x$ ) *borel* ( $T \ x$ ))) *differentiable at*  $t \ t > 0$  **for**  $t::\text{real}$   
 ⟨proof⟩

**lemma** *deriv-t-p-mu*:  $\text{deriv } (\lambda s. \mathbb{P}\{s \& x\}) \ t = - \mathbb{P}\{t \& x\} * \mathbb{M}\text{-}(x+t)$   
**if** ( $\lambda s. \mathbb{P}\{s \& x\}$ ) *differentiable at*  $t \ t > 0$  **for**  $t::\text{real}$   
 ⟨proof⟩

**lemma** *pdfTx-p-mu-AE*: *AE*  $s$  *in* *lborel*.  $s > 0 \longrightarrow \text{pdfT } x \ s = \mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s)$   
 ⟨proof⟩

**lemma** *LBINT-p-mu-q-defer*: (*LBINT*  $s:\{f < ..f+t\}$ .  $\mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s)$ ) =  $\mathbb{Q}\{f | t \& x\}$   
**if**  $t \geq 0$   $f \geq 0$  **for**  $t \ f :: \text{real}$   
 ⟨proof⟩

**corollary** *LBINT-p-mu-q*: (*LBINT*  $s:\{0 < ..t\}$ .  $\mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s)$ ) =  $\mathbb{Q}\{t \& x\}$   
**if**  $t \geq 0$  **for**  $t::\text{real}$   
 ⟨proof⟩

**lemma** *set-integrable-p-mu*: *set-integrable* *lborel*  $\{f < ..f+t\}$  ( $\lambda s. \mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s)$ )  
**if**  $t \geq 0$   $f \geq 0$  **for**  $t \ f :: \text{real}$   
 ⟨proof⟩

**lemma** *p-mu-has-integral-q-defer-Ioc*:  
 ( $\lambda s. \mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s)$ ) *has-integral*  $\mathbb{Q}\{f | t \& x\}$   $\{f < ..f+t\}$   
**if**  $t \geq 0$   $f \geq 0$  **for**  $t \ f :: \text{real}$   
 ⟨proof⟩

**lemma** *p-mu-has-integral-q-defer-Icc*:  
 ( $\lambda s. \mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s)$ ) *has-integral*  $\mathbb{Q}\{f | t \& x\}$   $\{f ..f+t\}$  **if**  $t \geq 0$   $f \geq 0$  **for**  
 $t \ f :: \text{real}$   
 ⟨proof⟩

**corollary** *p-mu-has-integral-q-Icc*:  
 ( $\lambda s. \mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s)$ ) *has-integral*  $\mathbb{Q}\{t \& x\}$   $\{0 ..t\}$  **if**  $t \geq 0$  **for**  $t::\text{real}$   
 ⟨proof⟩

**corollary** *p-mu-integrable-on-Icc*:  
 ( $\lambda s. \mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s)$ ) *integrable-on*  $\{0 ..t\}$  **if**  $t \geq 0$  **for**  $t::\text{real}$   
 ⟨proof⟩

**lemma** *e-ennreal-p-mu*:  $(\int^{+\xi} \text{ennreal } (T \ x \ \xi) \ \partial(\mathfrak{M} \downarrow \text{alive } x)) =$   
 $(\int^{+s \in \{0 ..\}} \text{ennreal } (\mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s) * s) \ \partial \text{lborel})$   
 ⟨proof⟩

**lemma** *e-LBINT-p-mu*:  $\mathbb{E}' \circ x = (\text{LBINT } s:\{0 ..\}. \mathbb{P}\{s \& x\} * \mathbb{M}\text{-}(x+s) * s)$

— Note that  $0 = 0$  holds when the life expectation diverges.  
 ⟨proof⟩

**lemma** *e-integral-p-mu*:  $\$e^{\circ-x} = \text{integral } \{0..\} (\lambda s. \$p\{-s\&x\} * \$\mu\{-(x+s)\} * s)$

— Note that  $0 = 0$  holds when the life expectation diverges.  
 ⟨proof⟩

**end**

**lemma** *p-has-real-derivative-x-cdfX*:

$((\lambda y. \$p\{-t\&y\}) \text{ has-real-derivative}$

$((\text{deriv } s\text{vl } (x+t) * s\text{vl } x - s\text{vl } (x+t) * \text{deriv } s\text{vl } x) / (s\text{vl } x)^2)) \text{ (at } x)$

**if**  $s\text{vl} \equiv \text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X) \text{ s\text{vl} differentiable at } x \text{ s\text{vl} differentiable at } (x+t)$

$t \geq 0 \ x < \$\psi \text{ for } x \ t :: \text{real}$

⟨proof⟩

**lemma** *p-has-real-derivative-x-p-mu*:

$((\lambda y. \$p\{-t\&y\}) \text{ has-real-derivative } \$p\{-t\&x\} * (\$mu\{-x\} - \$mu\{-(x+t)\})) \text{ (at } x)$

**if**  $\text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X) \text{ differentiable at } x \ \text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X) \text{ differentiable at } (x+t)$

$t \geq 0 \ x < \$\psi \text{ for } x \ t :: \text{real}$

⟨proof⟩

**corollary** *deriv-x-p-mu*:  $\text{deriv } (\lambda y. \$p\{-t\&y\}) \ x = \$p\{-t\&x\} * (\$mu\{-x\} - \$mu\{-(x+t)\})$

**if**  $\text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X) \text{ differentiable at } x \ \text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X) \text{ differentiable at } (x+t)$

$t \geq 0 \ x < \$\psi \text{ for } x \ t :: \text{real}$

⟨proof⟩

**lemma** *e-has-derivative-mu-e*:  $((\lambda x. \$e^{\circ-x}) \text{ has-real-derivative } (\$mu\{-x\} * \$e^{\circ-x} - 1)) \text{ (at } x)$

**if**  $\bigwedge x. x \in \{a <..< b\} \implies \text{set-integrable } \text{l borel } \{x..\} \ (\text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X))$

$\text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X) \text{ differentiable at } x \ x \in \{a <..< b\} \ b \leq \$\psi$

**for**  $a \ b \ x :: \text{real}$

⟨proof⟩

**corollary** *e-has-derivative-mu-e'*:  $((\lambda x. \$e^{\circ-x}) \text{ has-real-derivative } (\$mu\{-x\} * \$e^{\circ-x} - 1)) \text{ (at } x)$

**if**  $\bigwedge x. x \in \{a <..< b\} \implies \text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X) \text{ integrable-on } \{x..\}$

$\text{cdf } (\text{distr } \mathfrak{M} \text{ borel } X) \text{ differentiable at } x \ x \in \{a <..< b\} \ b \leq \$\psi$

**for**  $a \ b \ x :: \text{real}$

⟨proof⟩

## 5.2.11 Properties of Curtate Life Expectation

**context**

**fixes**  $x :: \text{real}$

**assumes**  $x\text{-lt-psi}[simp]: x < \$\psi$

**begin**

**lemma** *isCont-p-nat*:  $isCont (\lambda t. \mathbb{P}\{t \& x\}) (k + (1 :: real))$  **for**  $k :: nat$   
 ⟨proof⟩

**lemma** *curt-e-sum-p-smooth*:  $\mathbb{E}e-x = (\sum k. \mathbb{P}\{k+1 \& x\})$  **if** *summable*  $(\lambda k. \mathbb{P}\{k+1 \& x\})$   
 ⟨proof⟩

**lemma** *curt-e-rec-smooth*:  $\mathbb{E}e-x = \mathbb{P}x * (1 + \mathbb{E}e-(x+1))$  **if** *summable*  $(\lambda k. \mathbb{P}\{k+1 \& x\})$   
 $x+1 < \mathbb{P}\psi$   
 ⟨proof⟩

**lemma** *curt-e-sum-p-finite-smooth*:  $\mathbb{E}e-x = (\sum k < n. \mathbb{P}\{k+1 \& x\})$  **if**  $x+n+1 > \mathbb{P}\psi$  **for**  $n :: nat$   
 ⟨proof⟩

**lemma** *temp-curt-e-sum-p-smooth*:  $\mathbb{E}e-\{x:n\} = (\sum k < n. \mathbb{P}\{k+1 \& x\})$  **for**  $n :: nat$   
 ⟨proof⟩

**lemma** *temp-curt-e-rec-smooth*:  $\mathbb{E}e-\{x:n\} = \mathbb{P}x * (1 + \mathbb{E}e-\{x+1:n-1\})$   
**if**  $x+1 < \mathbb{P}\psi$   $n \neq 0$  **for**  $n :: nat$   
 ⟨proof⟩

**end**

**end**

### 5.3 Limited Survival Function

**locale** *limited-survival-function* = *survival-model* +  
**assumes** *psi-limited[simp]*:  $\mathbb{P}\psi < \infty$   
**begin**

**definition** *ult-age* ::  $nat$  ( $\mathbb{P}\omega$ )  
**where**  $\mathbb{P}\omega \equiv LEAST x :: nat. cdf (distr \mathfrak{M} borel X) x = 0$   
 — the conventional notation for ultimate age

**lemma** *cdfX-ceil-psi-0*:  $cdf (distr \mathfrak{M} borel X) \lceil real-of-ereal \mathbb{P}\psi \rceil = 0$   
 ⟨proof⟩

**lemma** *cdfX-omega-0*:  $cdf (distr \mathfrak{M} borel X) \mathbb{P}\omega = 0$   
 ⟨proof⟩

**corollary** *psi-le-omega*:  $\mathbb{P}\psi \leq \mathbb{P}\omega$   
 ⟨proof⟩

**corollary** *omega-pos*:  $\mathbb{P}\omega > 0$   
 ⟨proof⟩

**lemma** *omega-ceil-psi*:  $\mathbb{P}\omega = \lceil real-of-ereal \mathbb{P}\psi \rceil$



*<proof>*

**lemma** *ccdfX-0-equiv-nat*:  $ccdf (distr \mathfrak{M} \text{ borel } X) x = 0 \iff x \geq \$\omega$  **for**  $x::nat$   
*<proof>*

**lemma** *psi-le-iff-omega-le*:  $\psi \leq x \iff \$\omega \leq x$  **for**  $x::nat$   
*<proof>*

**context**

**fixes**  $x::nat$

**assumes**  $x\text{-lt-}\omega[simp]$ :  $x < \$\omega$

**begin**

**lemma**  $x\text{-lt-}\psi[simp]$ :  $x < \psi$   
*<proof>*

**lemma**  $p\text{-}0\text{-}1\text{-}nat$ :  $\$p\text{-}\{0\&x\} = 1$   
*<proof>*

**lemma**  $p\text{-}0\text{-}equiv\text{-}nat$ :  $\$p\text{-}\{t\&x\} = 0 \iff x+t \geq \$\omega$  **for**  $t::nat$   
*<proof>*

**lemma**  $q\text{-}0\text{-}0\text{-}nat$ :  $\$q\text{-}\{0\&x\} = 0$   
*<proof>*

**lemma**  $q\text{-}1\text{-}equiv\text{-}nat$ :  $\$q\text{-}\{t\&x\} = 1 \iff x+t \geq \$\omega$  **for**  $t::nat$   
*<proof>*

**lemma**  $q\text{-}defer\text{-}old\text{-}0\text{-}nat$ :  $\$q\text{-}\{f|t\&x\} = 0$  **if**  $\$ \omega \leq x+f$  **for**  $f t :: nat$   
*<proof>*

**lemma**  $curt\text{-}e\text{-}sum\text{-}P\text{-}finite\text{-}nat$ :  $\$e\text{-}x = (\sum k < n. \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \geq k + 1 \mid T x \xi > 0))$   
**if**  $x+n \geq \$\omega$  **for**  $n::nat$   
*<proof>*

**lemma**  $curt\text{-}e\text{-}sum\text{-}p\text{-}finite\text{-}nat$ :  $\$e\text{-}x = (\sum k < n. \$p\text{-}\{k+1\&x\})$   
**if**  $\bigwedge k::nat. k < n \implies isCont (\lambda t. \$p\text{-}\{t\&x\})$  **(real**  $k + 1)$   $x+n \geq \$\omega$  **for**  $n::nat$   
*<proof>*

**end**

**lemma**  $q\text{-}\omega\text{-}1$ :  $\$q\text{-}(\$ \omega - 1) = 1$   
*<proof>*

**end**

**end**

**theory** *Life-Table*

**imports** *Survival-Model*  
**begin**

## 6 Life Table

Define a life table axiomatically.

**locale** *life-table* =  
**fixes**  $l :: \text{real} \Rightarrow \text{real}$  ( $\langle \$l' \rightarrow [101] 200 \rangle$ )  
**assumes** *l-0-pos*:  $0 < l\ 0$   
**and** *l-neg-nil*:  $\bigwedge x. x \leq 0 \implies l\ x = l\ 0$   
**and** *l-PInfty-0*: ( $l \longrightarrow 0$ ) *at-top*  
**and** *l-antimono*: *antimono*  $l$   
**and** *l-right-continuous*:  $\bigwedge x. \text{continuous } (\text{at-right } x) l$   
**begin**

### 6.1 Basic Properties of Life Table

**lemma** *l-0-neg-0[simp]*:  $\$l\ 0 \neq 0$   
 $\langle \text{proof} \rangle$

**lemma** *l-nonneg[simp]*:  $\$l\ x \geq 0$  **for**  $x :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma** *l-bounded[simp]*:  $\$l\ x \leq \$l\ 0$  **for**  $x :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma** *l-measurable[measurable, simp]*:  $l \in \text{borel-measurable borel}$   
 $\langle \text{proof} \rangle$

**lemma** *l-left-continuous-nonpos*: *continuous* (*at-left*  $x$ )  $l$  **if**  $x \leq 0$  **for**  $x :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma** *l-integrable-Icc*: *set-integrable lborel*  $\{a..b\}$   $l$  **for**  $a\ b :: \text{real}$   
 $\langle \text{proof} \rangle$

**corollary** *l-integrable-on-Icc*:  $l$  *integrable-on*  $\{a..b\}$  **for**  $a\ b :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma** *l-integrable-Icc-shift*: *set-integrable lborel*  $\{a..b\}$   $(\lambda t. \$l\ (x+t))$  **for**  $a\ b\ x :: \text{real}$   
 $\langle \text{proof} \rangle$

**corollary** *l-integrable-on-Icc-shift*:  $(\lambda t. \$l\ (x+t))$  *integrable-on*  $\{a..b\}$  **for**  $x\ a\ b :: \text{real}$   
 $\langle \text{proof} \rangle$

**lemma** *l-normal-antimono*: *antimono*  $(\lambda x. \$l\ x / \$l\ 0)$   
 $\langle \text{proof} \rangle$

**lemma** *compl-l-normal-right-continuous*: continuous (at-right  $x$ )  $(\lambda x. 1 - \mathbb{N}l-x / \mathbb{N}l-0)$  for  $x::real$

*<proof>*

**lemma** *compl-l-normal-NInfty-0*:  $((\lambda x. 1 - \mathbb{N}l-x / \mathbb{N}l-0) \longrightarrow 0)$  at-bot

*<proof>*

**lemma** *compl-l-normal-PInfty-1*:  $((\lambda x. 1 - \mathbb{N}l-x / \mathbb{N}l-0) \longrightarrow 1)$  at-top

*<proof>*

**lemma** *compl-l-real-distribution*: real-distribution (interval-measure  $(\lambda x. 1 - \mathbb{N}l-x / \mathbb{N}l-0)$ )

*<proof>*

**definition** *total* ::  $real \Rightarrow real$  ( $\langle \mathbb{N}T'-\rangle$  [101] 200) **where**  $\mathbb{N}T-x \equiv LBINT y:\{x..\}$ .  $\mathbb{N}l-y$

— the number of lives older than the ones aged  $x$

— The parameter  $x$  must be nonnegative.

**lemma** *T-nonneg[simp]*:  $\mathbb{N}T-x \geq 0$  for  $x::real$

*<proof>*

**definition** *total-finite*  $\equiv$  set-integrable lborel  $\{0..\}$   $l$

**lemma** *total-finite-iff-set-integrable-Ici*:

$total-finite \longleftrightarrow set-integrable\ lborel\ \{x..\}\ l$  for  $x::real$

*<proof>*

**lemma** *total-finite-iff-integrable-on-Ici*:  $total-finite \longleftrightarrow l$  integrable-on  $\{x..\}$  for  $x::real$

*<proof>*

**lemma** *total-finite-iff-summable*:  $total-finite \longleftrightarrow summable\ (\lambda k. \mathbb{N}l-(x+k))$  for  $x::real$

*<proof>*

**lemma** *T-tendsto-0*:  $((\lambda x. \mathbb{N}T-x) \longrightarrow 0)$  at-top **if**  $total-finite$

*<proof>*

**definition** *lives* ::  $real \Rightarrow real \Rightarrow real$  ( $\langle \mathbb{N}L'-\{\&x-\} \rangle$  [0,0] 200)

**where**  $\mathbb{N}L-\{n\&x\} \equiv LBINT y:\{x..x+n\}$ .  $\mathbb{N}l-y$

— the number of lives between ages  $x$  and  $x+n$

— The parameter  $x$  must be nonnegative.

— The parameter  $n$  is usually nonnegative, but theoretically it can be negative.

**abbreviation** *lives-1* ::  $real \Rightarrow real$  ( $\langle \mathbb{N}L'-\rangle$  [101] 200)

**where**  $\mathbb{N}L-x \equiv \mathbb{N}L-\{1\&x\}$

**lemma** *l-has-integral-L*:  $(l$  has-integral  $\mathbb{N}L-\{n\&x\})\ \{x..x+n\}$  for  $x\ n::real$

*<proof>*

**lemma** *L-neg-0[simp]*:  $\$L\{-n\&x\} = 0$  **if**  $n < 0$  **for**  $x n :: real$   
*<proof>*

**lemma** *L-nonneg[simp]*:  $\$L\{n\&x\} \geq 0$  **for**  $x n :: real$   
*<proof>*

**lemma** *L-T*:  $\$L\{-n\&x\} = \$T-x - \$T\{-x+n\}$  **if** *total-finite*  $n \geq 0$  **for**  $x n :: real$   
*<proof>*

**lemma** *L-sums-T*:  $(\lambda k. \$L\{-x+k\})$  *sums*  $\$T-x$  **if** *total-finite* **for**  $x::real$   
*<proof>*

**definition** *death* ::  $real \Rightarrow real \Rightarrow real$  ( $\langle \$d'\{-\&- \} \rangle [0,0] 200$ )  
**where**  $\$d\{-t\&x\} \equiv \max 0 (\$l-x - \$l\{-x+t\})$   
— the number of deaths between ages  $x$  and  $x+t$   
— The parameter  $t$  is usually nonnegative, but theoretically it can be negative.

**abbreviation** *death1* ::  $real \Rightarrow real$  ( $\langle \$d'\{- \} \rangle [101] 200$ )  
**where**  $\$d-x \equiv \$d\{-1\&x\}$

**lemma** *death-def-nonneg*:  $\$d\{-t\&x\} = \$l-x - \$l\{-x+t\}$  **if**  $t \geq 0$  **for**  $t x :: real$   
*<proof>*

**lemma** *d-nonpos-0*:  $\$d\{-t\&x\} = 0$  **if**  $t \leq 0$  **for**  $t x :: real$   
*<proof>*

**corollary** *d-0-0*:  $\$d\{-0\&x\} = 0$  **for**  $x::real$   
*<proof>*

**lemma** *d-nonneg[simp]*:  $\$d\{-t\&x\} \geq 0$  **for**  $t x :: real$   
*<proof>*

**lemma** *dx-l*:  $\$d-x = \$l-x - \$l\{-x+1\}$  **for**  $x::real$   
*<proof>*

**lemma** *sum-dx-l*:  $(\sum k < n. \$d\{-x+k\}) = \$l-x - \$l\{-x+n\}$  **for**  $x::real$  **and**  $n::nat$   
*<proof>*

**corollary** *d-sums-l*:  $(\lambda k. \$d\{-x+k\})$  *sums*  $\$l-x$  **for**  $x::real$   
*<proof>*

**lemma** *add-d*:  $\$d\{-t\&x\} + \$d\{-t'\&x+t\} = \$d\{-t+t'\&x\}$  **if**  $t \geq 0$   $t' \geq 0$  **for**  $t$   
 $t' :: real$   
*<proof>*

**definition** *die-central* ::  $real \Rightarrow real \Rightarrow real$  ( $\langle \$m'\{-\&- \} \rangle [0,0] 200$ )  
**where**  $\$m\{-n\&x\} \equiv \$d\{-n\&x\} / \$L\{-n\&x\}$

— central death rate

**abbreviation** *die-central-1* :: *real*  $\Rightarrow$  *real* ( $\langle \$m' \rightarrow [101] 200 \rangle$ )  
**where**  $\$m-x \equiv \$m-\{1 \& x\}$

## 6.2 Construction of Survival Model from Life Table

**definition** *life-table-measure* :: *real measure* ( $\langle \mathfrak{M} \rangle$ )  
**where**  $\mathfrak{M} \equiv$  *interval-measure* ( $\lambda x. 1 - \$l-x / \$l-0$ )

**lemma** *prob-space-actuary-MM*: *prob-space-actuary*  $\mathfrak{M}$   
*\langle proof \rangle*

**definition** *survival-model-X* :: *real*  $\Rightarrow$  *real* ( $\langle X \rangle$ ) **where**  $X \equiv \lambda x. x$

**lemma** *survival-model-MM-X*: *survival-model*  $\mathfrak{M} X$   
*\langle proof \rangle*

**end**

**sublocale** *life-table*  $\subseteq$  *survival-model*  $\mathfrak{M} X$   
*\langle proof \rangle*

**context** *life-table*  
**begin**

**interpretation** *distrX-RD*: *real-distribution distr*  $\mathfrak{M}$  *borel X*  
*\langle proof \rangle*

### 6.2.1 Relations between Life Table and Survival Function for X

**lemma** *ccdfX-l-normal*: *ccdf* (*distr*  $\mathfrak{M}$  *borel X*) = ( $\lambda x. \$l-x / \$l-0$ )  
*\langle proof \rangle*

**corollary** *deriv-ccdfX-l*: *deriv* (*ccdf* (*distr*  $\mathfrak{M}$  *borel X*))  $x =$  *deriv*  $l x / \$l-0$   
**if**  $l$  *differentiable at x* **for**  $x::real$   
*\langle proof \rangle*

**notation** *death-pt* ( $\langle \$\psi \rangle$ )

**lemma** *l-0-equiv*:  $\$l-x = 0 \iff x \geq \$\psi$  **for**  $x::real$   
*\langle proof \rangle*

**lemma** *d-old-0*:  $\$d-\{t \& x\} = 0$  **if**  $x \geq \$\psi$   $t \geq 0$  **for**  $x t :: real$   
*\langle proof \rangle*

**lemma** *d-l-equiv*:  $\$d-\{t \& x\} = \$l-x \iff x+t \geq \$\psi$  **if**  $t \geq 0$  **for**  $x t :: real$   
*\langle proof \rangle*

**lemma** *continuous-ccdfX-l*: *continuous*  $F$  (*ccdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ))  $\longleftrightarrow$  *continuous*  $F$   $l$

**for**  $F ::$  *real filter*  
 $\langle$ *proof* $\rangle$

**lemma** *has-real-derivative-ccdfX-l*:

(*ccdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ) *has-real-derivative*  $D$ ) (*at*  $x$ )  $\longleftrightarrow$   
 $(l$  *has-real-derivative*  $\$l-0 * D$ ) (*at*  $x$ )

**for**  $D x ::$  *real*  
 $\langle$ *proof* $\rangle$

**corollary** *differentiable-ccdfX-l*:

*ccdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ) *differentiable* (*at*  $x$ )  $\longleftrightarrow$   $l$  *differentiable* (*at*  $x$ )

**for**  $D x ::$  *real*  
 $\langle$ *proof* $\rangle$

**lemma** *PX-l-normal*:  $\mathcal{P}(\xi$  *in*  $\mathfrak{M}. X \xi > x) = \$l-x / \$l-0$  **for**  $x ::$  *real*

$\langle$ *proof* $\rangle$

**lemma** *set-integrable-ccdfX-l*:

*set-integrable* *lborel*  $A$  (*ccdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ))  $\longleftrightarrow$  *set-integrable* *lborel*  $A$   $l$

**if**  $A \in$  *sets* *lborel* **for**  $A ::$  *real set*

$\langle$ *proof* $\rangle$

**corollary** *integrable-ccdfX-l*: *integrable* *lborel* (*ccdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ))  $\longleftrightarrow$  *integrable* *lborel*  $l$

$\langle$ *proof* $\rangle$

**lemma** *integrable-on-ccdfX-l*:

*ccdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ) *integrable-on*  $A$   $\longleftrightarrow$   $l$  *integrable-on*  $A$  **for**  $A ::$  *real set*

$\langle$ *proof* $\rangle$

## 6.2.2 Relations between Life Table and Cumulative Distributive Function for $X$

**lemma** *cdfX-l-normal*: *cdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ) =  $(\lambda x. 1 - \$l-x / \$l-0)$  **for**  $x ::$  *real*

$\langle$ *proof* $\rangle$

**lemma** *deriv-cdfX-l*: *deriv* (*cdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ))  $x = -$  *deriv*  $l x / \$l-0$

**if**  $l$  *differentiable* *at*  $x$  **for**  $x ::$  *real*

$\langle$ *proof* $\rangle$

**lemma** *continuous-cdfX-l*: *continuous*  $F$  (*cdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ))  $\longleftrightarrow$  *continuous*  $F$   $l$

**for**  $F ::$  *real filter*  
 $\langle$ *proof* $\rangle$

**lemma** *has-real-derivative-cdfX-l*:

(*cdf* (*distr*  $\mathfrak{M}$  *borel*  $X$ ) *has-real-derivative*  $D$ ) (*at*  $x$ )  $\longleftrightarrow$

(*l* has-real-derivative  $- (\$l-0 * D) )$  (*at x*)  
**for** *D x* :: *real*  
 ⟨*proof*⟩

**lemma** *differentiable-cdfX-l*:  
*cdf (distr  $\mathfrak{M}$  borel X) differentiable (at x)  $\longleftrightarrow$  l differentiable (at x)* **for** *D x* ::  
*real*  
 ⟨*proof*⟩

**lemma** *PX-compl-l-normal*:  $\mathcal{P}(\xi \text{ in } \mathfrak{M}. X \xi \leq x) = 1 - \$l-x / \$l-0$  **for** *x*::*real*  
 ⟨*proof*⟩

### 6.2.3 Relations between Life Table and Survival Function for $T(x)$

**context**  
**fixes** *x*::*real*  
**assumes** *x-lt-psi[simp]*:  $x < \$\psi$   
**begin**

**notation** *futr-life* ( $\langle T \rangle$ )

**interpretation** *alivex-PS*: *prob-space  $\mathfrak{M}$  | alive x*  
 ⟨*proof*⟩

**interpretation** *distrTx-RD*: *real-distribution distr ( $\mathfrak{M}$  | alive x) borel (T x)* ⟨*proof*⟩

**lemma** *lx-neq-0[simp]*:  $\$l-x \neq 0$   
 ⟨*proof*⟩

**corollary** *lx-pos[simp]*:  $\$l-x > 0$   
 ⟨*proof*⟩

**lemma** *ccdfTx-l-normal*: *ccdf (distr ( $\mathfrak{M}$  | alive x) borel (T x)) t =  $\$l-(x+t) / \$l-x$*   
**if**  $t \geq 0$  **for** *t*::*real*  
 ⟨*proof*⟩

**lemma** *deriv-ccdfTx-l*:  
*deriv (ccdf (distr ( $\mathfrak{M}$  | alive x) borel (T x))) t = deriv ( $\lambda t. \$l-(x+t) / \$l-x$ ) t*  
**if**  $t > 0$  *l* differentiable at  $(x+t)$  **for** *t*::*real*  
 ⟨*proof*⟩

**lemma** *continuous-at-within-ccdfTx-l*:  
*continuous (at t within {0..}) (ccdf (distr ( $\mathfrak{M}$  | alive x) borel (T x)))  $\longleftrightarrow$*   
*continuous (at (x+t) within {x..}) l*  
**if**  $t \geq 0$  **for** *t*::*real*  
 ⟨*proof*⟩

**lemma** *isCont-ccdfTx-l*:

*isCont* (cdf (distr ( $\mathfrak{M} \downarrow$  alive  $x$ ) borel ( $T x$ )))  $t \longleftrightarrow$  *isCont*  $l(x+t)$  **if**  $t > 0$  **for**  $t::real$   
 ⟨proof⟩

**lemma** *has-real-derivative-cdfTx-l*:  
 (cdf (distr ( $\mathfrak{M} \downarrow$  alive  $x$ ) borel ( $T x$ )) *has-real-derivative*  $D$ ) (at  $t$ )  $\longleftrightarrow$   
 (*l has-real-derivative*  $\$l-x * D$ ) (at ( $x+t$ ))  
**if**  $t > 0$  **for**  $t D :: real$   
 ⟨proof⟩

**lemma** *differentiable-cdfTx-l*:  
 cdf (distr ( $\mathfrak{M} \downarrow$  alive  $x$ ) borel ( $T x$ )) *differentiable at*  $t \longleftrightarrow$   $l$  *differentiable* (at  
 ( $x+t$ ))  
**if**  $t > 0$  **for**  $t::real$   
 ⟨proof⟩

**lemma** *PTx-l-normal*:  $\mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi > t \mid T x \xi > 0) = \$l-(x+t) / \$l-x$  **if**  $t \geq$   
 $0$  **for**  $t::real$   
 ⟨proof⟩

#### 6.2.4 Relations between Life Table and Cumulative Distributive Function for $T(x)$

**lemma** *cdfTx-compl-l-normal*: cdf (distr ( $\mathfrak{M} \downarrow$  alive  $x$ ) borel ( $T x$ ))  $t = 1 -$   
 $\$l-(x+t) / \$l-x$   
**if**  $t \geq 0$  **for**  $t::real$   
 ⟨proof⟩

**lemma** *deriv-cdfTx-l*:  
 deriv (cdf (distr ( $\mathfrak{M} \downarrow$  alive  $x$ ) borel ( $T x$ )))  $t = -$  deriv ( $\lambda t. \$l-(x+t) / \$l-x$ )  $t$   
**if**  $t > 0$   $l$  *differentiable at* ( $x+t$ ) **for**  $t::real$   
 ⟨proof⟩

**lemma** *continuous-at-within-cdfTx-l*:  
 continuous (at  $t$  within  $\{0..\}$ ) (cdf (distr ( $\mathfrak{M} \downarrow$  alive  $x$ ) borel ( $T x$ )))  $\longleftrightarrow$   
 continuous (at ( $x+t$ ) within  $\{x..\}$ )  $l$   
**if**  $t \geq 0$  **for**  $t::real$   
 ⟨proof⟩

**lemma** *isCont-cdfTx-l*:  
*isCont* (cdf (distr ( $\mathfrak{M} \downarrow$  alive  $x$ ) borel ( $T x$ )))  $t \longleftrightarrow$  *isCont*  $l(x+t)$  **if**  $t > 0$  **for**  
 $t::real$   
 ⟨proof⟩

**lemma** *has-real-derivative-cdfTx-l*:  
 (cdf (distr ( $\mathfrak{M} \downarrow$  alive  $x$ ) borel ( $T x$ )) *has-real-derivative*  $D$ ) (at  $t$ )  $\longleftrightarrow$   
 (*l has-real-derivative*  $-\$l-x * D$ ) (at ( $x+t$ ))  
**if**  $t > 0$  **for**  $t D :: real$   
 ⟨proof⟩



**lemma** *differentiable-cdfTx-l:*

*cdf (distr (M | alive x) borel (T x)) differentiable at t  $\longleftrightarrow$  l differentiable (at (x+t))*  
**if**  $t > 0$  **for**  $t::real$   
*<proof>*

**lemma** *PTx-compl-l-normal:*  $\mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \leq t \mid T x \xi > 0) = 1 - \$l-(x+t) / \$l-x$

**if**  $t \geq 0$  **for**  $t::real$   
*<proof>*

## 6.2.5 Life Table and Actuarial Notations

**notation** *survive* ( $\langle \$p'\{-\&- \} \rangle [0,0]$  200)

**notation** *survive-1* ( $\langle \$p'\{-\} \rangle [101]$  200)

**notation** *die* ( $\langle \$q'\{-\&- \} \rangle [0,0]$  200)

**notation** *die-1* ( $\langle \$q'\{-\} \rangle [101]$  200)

**notation** *die-defer* ( $\langle \$q'\{-|\&- \} \rangle [0,0,0]$  200)

**notation** *die-defer-1* ( $\langle \$q'\{-|\&- \} \rangle [0,0]$  200)

**notation** *life-expect* ( $\langle \$e'\circ'\{-\} \rangle [101]$  200)

**notation** *temp-life-expect* ( $\langle \$e'\circ'\{-:- \} \rangle [0,0]$  200)

**notation** *curt-life-expect* ( $\langle \$e'\{-\} \rangle [101]$  200)

**notation** *temp-curt-life-expect* ( $\langle \$e'\{-:- \} \rangle [0,0]$  200)

**lemma** *p-l:*  $\$p\{-t\&x\} = \$l-(x+t) / \$l-x$  **if**  $t \geq 0$  **for**  $t::real$   
*<proof>*

**corollary** *p-1-l:*  $\$p-x = \$l-(x+1) / \$l-x$   
*<proof>*

**lemma** *isCont-p-l:*  $isCont (\lambda s. \$p\{-s\&x\}) t \longleftrightarrow isCont l (x+t)$  **if**  $t > 0$  **for**  $t::real$   
*<proof>*

**lemma** *total-finite-iff-p-set-integrable-Ici:*

*total-finite  $\longleftrightarrow$  set-integrable lborel {0..} ( $\lambda t. \$p\{-t\&x\}$ )*  
*<proof>*

**lemma** *p-PTx-ge-l-isCont:*  $\$p\{-t\&x\} = \mathcal{P}(\xi \text{ in } \mathfrak{M}. T x \xi \geq t \mid T x \xi > 0)$   
**if**  $isCont l (x+t)$   $t > 0$  **for**  $t::real$   
*<proof>*

**lemma** *q-defer-l:*  $\$q\{-f|t\&x\} = (\$l-(x+f) - \$l-(x+f+t)) / \$l-x$  **if**  $f \geq 0$   $t \geq 0$  **for**  $f t :: real$   
*<proof>*

**corollary** *q-defer-d-l:*  $\$q\{-f|t\&x\} = \$d\{-t \& x+f\} / \$l-x$  **if**  $f \geq 0$   $t \geq 0$  **for**  $f t :: real$   
*<proof>*

**corollary** *q-defer-1-d-l*:  $\$q\{-f\&x\} = \$d\{-x+f\} / \$l\text{-}x$  **if**  $f \geq 0$  **for**  $f::\text{real}$   
 ⟨*proof*⟩

**lemma** *q-d-l*:  $\$q\{-t\&x\} = \$d\{-t\&x\} / \$l\text{-}x$  **for**  $t::\text{real}$   
 ⟨*proof*⟩

**corollary** *q-1-d-l*:  $\$q\text{-}x = \$d\text{-}x / \$l\text{-}x$   
 ⟨*proof*⟩

**lemma** *LBINT-p-l*:  $(LBINT\ t:A.\ \$p\{-t\&x\}) = (LBINT\ t:A.\ \$l\{-x+t\}) / \$l\text{-}x$   
**if**  $A \subseteq \{0..\}$   $A \in \text{sets lborel}$  **for**  $A :: \text{real set}$   
 — Note that  $0 = 0$  holds when the integral diverges.  
 ⟨*proof*⟩

**corollary** *e-LBINT-l*:  $\$e'\circ\text{-}x = (LBINT\ t:\{0..\}\ \$l\{-x+t\}) / \$l\text{-}x$   
 — Note that  $0 = 0$  holds when the integral diverges.  
 ⟨*proof*⟩

**corollary** *e-LBINT-l-Icc*:  $\$e'\circ\text{-}x = (LBINT\ t:\{0..n\}.\ \$l\{-x+t\}) / \$l\text{-}x$  **if**  $x+n \geq$   
 $\psi$  **for**  $n::\text{real}$   
 ⟨*proof*⟩

**lemma** *temp-e-LBINT-l*:  $\$e'\circ\{-x:n\} = (LBINT\ t:\{0..n\}.\ \$l\{-x+t\}) / \$l\text{-}x$  **if**  $n \geq$   
 $0$  **for**  $n::\text{real}$   
 ⟨*proof*⟩

**lemma** *integral-p-l*:  $\text{integral } A (\lambda t.\ \$p\{-t\&x\}) = (\text{integral } A (\lambda t.\ \$l\{-x+t\})) / \$l\text{-}x$   
**if**  $A \subseteq \{0..\}$   $A \in \text{sets lborel}$  **for**  $A :: \text{real set}$   
 — Note that  $0 = 0$  holds when the integral diverges.  
 ⟨*proof*⟩

**corollary** *e-integral-l*:  $\$e'\circ\text{-}x = \text{integral } \{0..\} (\lambda t.\ \$l\{-x+t\}) / \$l\text{-}x$   
 — Note that  $0 = 0$  holds when the integral diverges.  
 ⟨*proof*⟩

**corollary** *e-integral-l-Icc*:  
 $\$e'\circ\text{-}x = \text{integral } \{0..n\} (\lambda t.\ \$l\{-x+t\}) / \$l\text{-}x$  **if**  $x+n \geq \psi$  **for**  $n::\text{real}$   
 ⟨*proof*⟩

**lemma** *e-pos-total-finite*:  $\$e'\circ\text{-}x > 0$  **if** *total-finite*  
 ⟨*proof*⟩

**lemma** *temp-e-integral-l*:  
 $\$e'\circ\{-x:n\} = \text{integral } \{0..n\} (\lambda t.\ \$l\{-x+t\}) / \$l\text{-}x$  **if**  $n \geq 0$  **for**  $n::\text{real}$   
 ⟨*proof*⟩

**lemma** *curt-e-sum-l*:  $\$e\text{-}x = (\sum k.\ \$l\{-x+k+1\}) / \$l\text{-}x$  **if** *total-finite*  $\wedge k::\text{nat. is-Cont } l (x+k+1)$

*<proof>*

**lemma** *curt-e-sum-l-finite*:  $\$e\text{-}x = (\sum_{k < n} \$l\text{-}(x+k+1)) / \$l\text{-}x$   
**if**  $\bigwedge k :: \text{nat}. k < n \implies \text{isCont } l \text{ } (x+k+1) \text{ } x+n+1 > \$\psi$  **for**  $n :: \text{nat}$   
*<proof>*

**lemma** *temp-curt-e-sum-p*:  $\$e\text{-}\{x:n\} = (\sum_{k < n} \$l\text{-}(x+k+1)) / \$l\text{-}x$   
**if**  $\bigwedge k :: \text{nat}. k < n \implies \text{isCont } l \text{ } (x+k+1)$  **for**  $n :: \text{nat}$   
*<proof>*

**lemma** *e-T-l*:  $\$e\text{'o-}x = \$T\text{-}x / \$l\text{-}x$   
*<proof>*

**lemma** *temp-e-L-l*:  $\$e\text{'o-}\{x:n\} = \$L\text{-}\{n\&x\} / \$l\text{-}x$  **if**  $n \geq 0$  **for**  $n :: \text{real}$   
*<proof>*

**lemma** *m-q-e*:  $\$m\text{-}\{n\&x\} = \$q\text{-}\{n\&x\} / \$e\text{'o-}\{x:n\}$  **if**  $n \geq 0$  **for**  $n :: \text{real}$   
*<proof>*

**end**

**lemma** *l-p*:  $\$l\text{-}x / \$l\text{-}0 = \$p\text{-}\{x\&0\}$  **for**  $x :: \text{real}$   
*<proof>*

**lemma** *e-p-e-total-finite*:  $\$e\text{'o-}x = \$e\text{'o-}\{x:n\} + \$p\text{-}\{n\&x\} * \$e\text{'o-}(x+n)$   
**if** *total-finite*  $n \geq 0 \text{ } x+n < \$\psi$  **for**  $x \text{ } n :: \text{real}$   
*<proof>*

**proposition** *x-ex-const-equiv-total-finite*:  $x + \$e\text{'o-}x = y + \$e\text{'o-}y \iff \$q\text{-}\{y-x\&x\} = 0$   
**if** *total-finite*  $x \leq y \text{ } y < \$\psi$  **for**  $x \text{ } y :: \text{real}$   
*<proof>*

**corollary** *x-ex-const-iff-l-const*:  $x + \$e\text{'o-}x = y + \$e\text{'o-}y \iff \$l\text{-}x = \$l\text{-}y$   
**if** *total-finite*  $x \leq y \text{ } y < \$\psi$  **for**  $x \text{ } y :: \text{real}$   
*<proof>*

**end**

### 6.3 Piecewise Differentiable Life Table

**locale** *smooth-life-table = life-table +*  
**assumes** *l-piecewise-differentiable[simp]*: *l piecewise-differentiable-on UNIV*  
**begin**

**lemma** *smooth-survival-function-MM-X*: *smooth-survival-function*  $\mathfrak{M} X$   
*<proof>*

**end**

**sublocale** *smooth-life-table*  $\subseteq$  *smooth-survival-function*  $\mathfrak{M} X$   
 ⟨proof⟩

**context** *smooth-life-table*  
**begin**

**notation** *force-mortal* ( $\langle \$\mu' \rightarrow [101] 200 \rangle$ )

**lemma** *l-continuous[simp]*: *continuous-on UNIV l*  
 ⟨proof⟩

**lemma** *l-nondifferentiable-finite-set[simp]*: *finite  $\{x. \neg l \text{ differentiable at } x\}$*   
 ⟨proof⟩

**lemma** *l-differentiable-borel-set[measurable, simp]*:  $\{x. l \text{ differentiable at } x\} \in \text{sets borel}$   
 ⟨proof⟩

**lemma** *l-differentiable-AE*: *AE x in lborel. l differentiable at x*  
 ⟨proof⟩

**lemma** *deriv-l-measurable[measurable]*: *deriv l  $\in$  borel-measurable borel*  
 ⟨proof⟩

**lemma** *pdfX-l-normal*:  
*pdfX x = (if l differentiable at x then - deriv l x / \$l-0 else 0) for x::real*  
 ⟨proof⟩

**lemma** *mu-deriv-l*:  $\$\mu-x = - \text{deriv } l \ x / \$l-x$  **if** *l differentiable at x* **for** *x::real*  
 ⟨proof⟩

**lemma** *mu-nonneg-differentiable-l*:  $\$\mu-x \geq 0$  **if** *l differentiable at x* **for** *x::real*  
 ⟨proof⟩

**lemma** *mu-deriv-ln-l*:  
 $\$\mu-x = - \text{deriv } (\lambda x. \ln (\$l-x)) \ x$  **if** *l differentiable at x*  $x < \$\psi$  **for** *x::real*  
 ⟨proof⟩

**lemma** *deriv-l-shift*:  $\text{deriv } l \ (x+t) = \text{deriv } (\lambda t. \$l-(x+t)) \ t$   
**if** *l differentiable at (x+t)* **for** *x t :: real*  
 ⟨proof⟩

**context**  
**fixes** *x::real*  
**assumes** *x-lt-psi[simp]*:  $x < \$\psi$   
**begin**

**lemma** *p-mu-l*:  $\$p-\{t\&x\} * \$\mu-(x+t) = - \text{deriv } l \ (x+t) / \$l-x$

**if**  $l$  differentiable at  $(x+t)$   $t > 0$   $x+t < \psi$  **for**  $t::real$   
 <proof>

**lemma**  $p\text{-}\mu\text{-}l\text{-}AE$ :  $AE$   $s$  in  $lborel$ .  $0 < s \wedge x+s < \psi \longrightarrow \mathbb{P}\{s \& x\} * \mathbb{P}\mu\text{-}(x+s)$   
 $= - \text{deriv } l(x+s) / \mathbb{P}l\text{-}x$   
 <proof>

**lemma**  $LBINT\text{-}l\text{-}\mu\text{-}q$ :  $(LBINT\ s:\{f < ..f+t\}. \mathbb{P}l\text{-}(x+s) * \mathbb{P}\mu\text{-}(x+s)) / \mathbb{P}l\text{-}x = \mathbb{P}q\text{-}\{f|t \& x\}$   
**if**  $t \geq 0$   $f \geq 0$  **for**  $t\ f :: real$   
 <proof>

**lemma**  $set\text{-}integrable\text{-}l\text{-}\mu$ :  $set\text{-}integrable\ lborel\ \{f < ..f+t\}$   $(\lambda s. \mathbb{P}l\text{-}(x+s) * \mathbb{P}\mu\text{-}(x+s))$   
**if**  $t \geq 0$   $f \geq 0$  **for**  $t\ f :: real$   
 <proof>

**lemma**  $l\text{-}\mu\text{-}has\text{-}integral\text{-}q\text{-}defer$ :  
 $((\lambda s. \mathbb{P}l\text{-}(x+s) * \mathbb{P}\mu\text{-}(x+s) / \mathbb{P}l\text{-}x)$   $has\text{-}integral\ \mathbb{P}q\text{-}\{f|t \& x\})$   $\{f..f+t\}$   
**if**  $t \geq 0$   $f \geq 0$  **for**  $t\ f :: real$   
 <proof>

**corollary**  $l\text{-}\mu\text{-}has\text{-}integral\text{-}q$ :  
 $((\lambda s. \mathbb{P}l\text{-}(x+s) * \mathbb{P}\mu\text{-}(x+s) / \mathbb{P}l\text{-}x)$   $has\text{-}integral\ \mathbb{P}q\text{-}\{t \& x\})$   $\{0..t\}$  **if**  $t \geq 0$  **for**  $t::real$   
 <proof>

**lemma**  $l\text{-}\mu\text{-}has\text{-}integral\text{-}d$ :  
 $((\lambda s. \mathbb{P}l\text{-}(x+s) * \mathbb{P}\mu\text{-}(x+s))$   $has\text{-}integral\ \mathbb{P}d\text{-}\{t \& x+f\})$   $\{f..f+t\}$   
**if**  $t \geq 0$   $f \geq 0$  **for**  $t\ f :: real$   
 <proof>

**corollary**  $l\text{-}\mu\text{-}has\text{-}integral\text{-}d\text{-}1$ :  
 $((\lambda s. \mathbb{P}l\text{-}(x+s) * \mathbb{P}\mu\text{-}(x+s))$   $has\text{-}integral\ \mathbb{P}d\text{-}(x+f))$   $\{f..f+1\}$  **if**  $t \geq 0$   $f \geq 0$  **for**  $t$   
 $f :: real$   
 <proof>

**lemma**  $e\text{-}LBINT\text{-}l$ :  $\mathbb{P}e^{\circ-x} = (LBINT\ s:\{0..\}. \mathbb{P}l\text{-}(x+s) * \mathbb{P}\mu\text{-}(x+s) * s) / \mathbb{P}l\text{-}x$   
 — Note that  $0 = 0$  holds when the life expectation diverges.  
 <proof>

**lemma**  $e\text{-}integral\text{-}l$ :  $\mathbb{P}e^{\circ-x} = integral\ \{0..\}$   $(\lambda s. \mathbb{P}l\text{-}(x+s) * \mathbb{P}\mu\text{-}(x+s) * s) / \mathbb{P}l\text{-}x$   
 — Note that  $0 = 0$  holds when the life expectation diverges.  
 <proof>

**lemma**  $m\text{-}LBINT\text{-}p\text{-}\mu$ :  $\mathbb{P}m\text{-}\{n \& x\} = (LBINT\ t:\{0 < ..n\}. \mathbb{P}p\text{-}\{t \& x\} * \mathbb{P}\mu\text{-}(x+t))$   
 $/ (LBINT\ t:\{0..n\}. \mathbb{P}p\text{-}\{t \& x\})$   
**if**  $n \geq 0$  **for**  $n::real$   
 <proof>

**lemma**  $m\text{-}integral\text{-}p\text{-}\mu$ :  
 $\mathbb{P}m\text{-}\{n \& x\} = integral\ \{0..n\}$   $(\lambda t. \mathbb{P}p\text{-}\{t \& x\} * \mathbb{P}\mu\text{-}(x+t)) / integral\ \{0..n\}$   $(\lambda t.$

$\$p\{-t&x\}$   
**if**  $n \geq 0$  **for**  $n::real$   
 $\langle proof \rangle$

**end**

**lemma** *deriv-x-p-mu-l*:  $deriv (\lambda y. \$p\{-t&y\}) x = \$p\{-t&x\} * (\$mu-x - \$mu-(x+t))$   
**if**  $l$  differentiable at  $x$   $l$  differentiable at  $(x+t)$   $t \geq 0$   $x < \$psi$  **for**  $x t :: real$   
 $\langle proof \rangle$

**lemma** *e-has-derivative-mu-e-l*:  $((\lambda x. \$e^{\circ}x)$  has-real-derivative  $(\$mu-x * \$e^{\circ}x - 1))$  (at  $x$ )  
**if** total-finite  $l$  differentiable at  $x$   $x \in \{a < .. < b\}$   $b \leq \$psi$  **for**  $a b x :: real$   
 $\langle proof \rangle$

**corollary** *e-has-derivative-mu-e-l'*:  $((\lambda x. \$e^{\circ}x)$  has-real-derivative  $(\$mu-x * \$e^{\circ}x - 1))$  (at  $x$ )  
**if** total-finite  $l$  differentiable at  $x$   $x \in \{a < .. < b\}$   $b \leq \$psi$  **for**  $a b x :: real$   
 $\langle proof \rangle$

**context**

**fixes**  $x::real$

**assumes**  $x-lt-psi[simp]$ :  $x < \$psi$

**begin**

**lemma** *curt-e-sum-l-smooth*:  $\$e-x = (\sum k. \$l(x+k+1)) / \$l-x$  **if** total-finite  
 $\langle proof \rangle$

**lemma** *curt-e-sum-l-finite-smooth*:  $\$e-x = (\sum k < n. \$l(x+k+1)) / \$l-x$  **if**  $x+n+1 > \$psi$  **for**  $n::nat$   
 $\langle proof \rangle$

**lemma** *temp-curt-e-sum-l-smooth*:  $\$e-\{x:n\} = (\sum k < n. \$l(x+k+1)) / \$l-x$  **for**  $n::nat$   
 $\langle proof \rangle$

**end**

**end**

## 6.4 Interpolations

**context** *life-table*

**begin**

**definition** *linear-interpolation*  $\equiv$

$\forall (x::nat)(t::real). 0 \leq t \wedge t \leq 1 \longrightarrow \$l(x+t) = (1-t)*\$l-x + t*\$l(x+1)$

**lemma** *linear-l*:  $\$l(x+t) = (1-t)*\$l-x + t*\$l(x+1)$

**if linear-interpolation**  $0 \leq t \leq 1$  **for**  $x::nat$  **and**  $t::real$   
 <proof>

**lemma linear-l-d:**  $\$l(x+t) = \$l-x - t*\$d-x$   
**if linear-interpolation**  $0 \leq t \leq 1$  **for**  $x::nat$  **and**  $t::real$   
 <proof>

**lemma linear-p-q:**  $\$p-\{t\&x\} = 1 - t*\$q-x$   
**if linear-interpolation**  $0 \leq t \leq 1$   $x < \$\psi$  **for**  $x::nat$  **and**  $t::real$   
 <proof>

**lemma linear-q:**  $\$q-\{t\&x\} = t*\$q-x$   
**if linear-interpolation**  $0 \leq t \leq 1$   $x < \$\psi$  **for**  $x::nat$  **and**  $t::real$   
 <proof>

**lemma linear-L-l-d:**  $\$L-x = \$l-x - \$d-x / 2$  **if linear-interpolation** **for**  $x::nat$   
 <proof>

**lemma linear-L-l-d':**  $\$L-x = \$l(x+1) + \$d-x / 2$  **if linear-interpolation** **for**  $x::nat$   
 <proof>

**lemma linear-l-continuous:** *continuous-on UNIV l* **if linear-interpolation**  
 <proof>

**lemma linear-l-sums-T-l:**  $(\lambda k. \$l(x + Suc k))$  *sums*  $(\$T-x - \$l-x / 2)$   
**if linear-interpolation total-finite** **for**  $x::nat$   
 <proof>

**corollary linear-T-suminf-l:**  $\$T-x = (\sum k. \$l(x+k+1)) + \$l-x / 2$   
**if linear-interpolation total-finite** **for**  $x::nat$   
 <proof>

**lemma linear-mx-q:**  $\$m-x = \$q-x / (1 - \$q-x / 2)$  **if linear-interpolation**  $x < \$\psi$   
**for**  $x::nat$   
 <proof>

**lemma linear-e-curt-e:**  $\$e\circ-x = \$e-x + 1/2$   
**if linear-interpolation total-finite**  $x < \$\psi$  **for**  $x::nat$   
 <proof>

**end**

**context smooth-life-table**  
**begin**

**lemma linear-l-has-derivative-at-frac:**  
 $((\lambda s. \$l(x+s))$  *has-real-derivative*  $- \$d-x$ ) (at  $t$ )  
**if linear-interpolation**  $0 < t < 1$  **for**  $x::nat$  **and**  $t::real$   
 <proof>

**lemma** *linear-l-has-derivative-at-frac'*:

(*l* has-real-derivative –  $\$d-x$ ) (at *y*)

**if** linear-interpolation  $x < y < x+1$  **for**  $x::nat$  **and**  $y::real$

*<proof>*

**lemma** *linear-l-differentiable-on-frac*:

*l* differentiable-on  $\{x <..<x+1\}$  **if** linear-interpolation **for**  $x::nat$

*<proof>*

**lemma** *linear-l-has-right-derivative-at-nat*:

(*l* has-real-derivative –  $\$d-x$ ) (at-right *x*) **if** linear-interpolation **for**  $x::nat$

*<proof>*

**lemma** *linear-l-has-left-derivative-at-nat*:

(*l* has-real-derivative –  $\$d-(real\ x - 1)$ ) (at-left *x*) **if** linear-interpolation **for**

$x::nat$

*<proof>*

**lemma** *linear-l-has-derivative-at-nat-iff-d*:

(*l* has-real-derivative –  $\$d-x$ ) (at *x*)  $\longleftrightarrow$   $\$d-x = \$d-(real\ x - 1)$

**if** linear-interpolation **for**  $x::nat$

*<proof>*

**lemma** *linear-l-differentiable-at-nat-iff-d*:

*l* differentiable at *x*  $\longleftrightarrow$   $\$d-x = \$d-(real\ x - 1)$

**if** linear-interpolation **for**  $x::nat$

*<proof>*

**lemma** *linear-l-limited*:  $\$ψ < ∞$  **if** linear-interpolation

*<proof>*

**lemma** *linear-mu-q*:  $\$μ-(x+t) = \$q-x / (1 - t*\$q-x)$

**if** linear-interpolation *l* differentiable at  $(x+t)$   $0 < t < 1$   $x+t < \$ψ$

**for**  $x::nat$  **and**  $t::real$

*<proof>*

**definition** *exponential-interpolation*  $\equiv$

$\forall (x::nat)(t::real). x+1 < \$ψ \longrightarrow 0 \leq t \wedge t < 1 \longrightarrow \$μ-(x+t) = \$μ-x$

— Without  $x+1 < \$ψ$ , the smooth life table could not be limited.

**lemma** *exponential-mu*:  $\$μ-(x+t) = \$μ-x$

**if** exponential-interpolation  $x+1 < \$ψ$   $0 \leq t < 1$  **for**  $x::nat$  **and**  $t::real$

*<proof>*

**corollary** *exponential-mu'*:  $\$μ-y = \$μ-x$

**if** exponential-interpolation  $x \leq y < x+1$   $x+1 < \$ψ$  **for**  $x::nat$  **and**  $y::real$

*<proof>*



**lemma** *exponential-integral-mu*:  $\text{integral } \{x..<x+t\} (\lambda y. \$\mu-y) = \$\mu-x * t$   
**if** *exponential-interpolation*  $x+1 < \$\psi$   $0 \leq t \leq 1$  **for**  $x::\text{nat}$  **and**  $t::\text{real}$   
 <proof>

**lemma** *exponential-p-mu*:  $\$p-x = \exp(-\$ \mu-x)$  **if** *exponential-interpolation*  $x+1 < \$\psi$  **for**  $x::\text{nat}$   
 <proof>

**corollary** *exponential-mu-p*:  $\$ \mu-x = -\ln(\$p-x)$  **if** *exponential-interpolation*  $x+1 < \$\psi$  **for**  $x::\text{nat}$   
 <proof>

**corollary** *exponential-mu-xt-p*:  $\$ \mu-(x+t) = -\ln(\$p-x)$   
**if** *exponential-interpolation*  $x+1 < \$\psi$   $0 \leq t < 1$  **for**  $x::\text{nat}$  **and**  $t::\text{real}$   
 <proof>

**corollary** *exponential-q-mu*:  $\$q-x = 1 - \exp(-\$ \mu-x)$   
**if** *exponential-interpolation*  $x+1 < \$\psi$  **for**  $x::\text{nat}$   
 <proof>

**lemma** *exponential-p*:  $\$p-\{t\&x\} = (\$p-x).\hat{t}$   
**if** *exponential-interpolation*  $x+1 < \$\psi$   $0 \leq t \leq 1$  **for**  $x::\text{nat}$  **and**  $t::\text{real}$   
 <proof>

**lemma** *exponential-q*:  $\$q-\{t\&x\} = 1 - (1 - \$q-x).\hat{t}$   
**if** *exponential-interpolation*  $x+1 < \$\psi$   $0 \leq t \leq 1$  **for**  $x::\text{nat}$  **and**  $t::\text{real}$   
 <proof>

**lemma** *exponential-l-p*:  $\$l-(x+t) = \$l-x * (\$p-x).\hat{t}$   
**if** *exponential-interpolation*  $x+1 < \$\psi$   $0 \leq t \leq 1$  **for**  $x::\text{nat}$  **and**  $t::\text{real}$   
 <proof>

**lemma** *exponential-l-has-derivative-at-frac*:  
 $((\lambda s. \$l-(x+s)) \text{ has-real-derivative } (-\$l-x * \$ \mu-x * (\$p-x).\hat{t})) (at t)$   
**if** *exponential-interpolation*  $x+1 < \$\psi$   $0 < t < 1$  **for**  $x::\text{nat}$  **and**  $t::\text{real}$   
 <proof>

**lemma** *exponential-l-has-derivative-at-frac'*:  
 $(l \text{ has-real-derivative } (-\$l-x * \$ \mu-x * (\$p-x).\hat{(y-x)})) (at y)$   
**if** *exponential-interpolation*  $x+1 < \$\psi$   $x < y < x+1$  **for**  $x::\text{nat}$  **and**  $y::\text{real}$   
 <proof>

**lemma** *exponential-l-differentiable-on-frac*:  
 $l$  *differentiable-on*  $\{x..<x+1\}$  **if** *exponential-interpolation*  $x+1 < \$\psi$  **for**  $x::\text{nat}$   
 <proof>

**lemma** *exponential-l-has-right-derivative-at-nat*:  
 $(l \text{ has-real-derivative } (-\$l-x * \$ \mu-x)) (at-right x)$   
**if** *exponential-interpolation*  $x+1 < \$\psi$  **for**  $x::\text{nat}$

*<proof>*

**lemma** *exponential-l-has-left-derivative-at-nat:*  
(*l* has-real-derivative ( $- \$l-x * \$\mu-(\text{real } x - 1)$ )) (*at-left* *x*)  
if exponential-interpolation  $x < \$\psi$  for  $x::\text{nat}$   
*<proof>*

**lemma** *exponential-l-has-derivative-at-nat-iff-mu:*  
(*l* has-real-derivative ( $- \$l-x * \$\mu-x$ )) (*at* *x*)  $\longleftrightarrow \$\mu-x = \$\mu-(\text{real } x - 1)$   
if exponential-interpolation  $x+1 < \$\psi$  for  $x::\text{nat}$   
*<proof>*

**lemma** *exponential-l-differentiable-at-nat-iff-mu:*  
*l* differentiable at *x*  $\longleftrightarrow \$\mu-x = \$\mu-(\text{real } x - 1)$   
if exponential-interpolation  $x+1 < \$\psi$  for  $x::\text{nat}$   
*<proof>*

**lemma** *exponential-L-d-mu:*  $\$L-x = \$d-x / \$\mu-x$   
if exponential-interpolation  $\$ \mu-x \neq 0$   $x+1 < \$\psi$  for  $x::\text{nat}$   
*<proof>*

**lemma** *exponential-mx-mu:*  $\$m-x = \$\mu-x$  if exponential-interpolation  $x+1 < \$\psi$   
for  $x::\text{nat}$   
*<proof>*

**lemma** *exponential-d-mu-sums-T:*  $(\lambda k. \$d-(x+k) / \$\mu-(x+k))$  sums  $\$T-x$   
if exponential-interpolation total-finite  $\bigwedge k::\text{nat}. \$\mu-(x+k) \neq 0$  for  $x::\text{nat}$   
*<proof>*

**lemma** *exponential-e-d-l-mu:*  $(\lambda k. \$d-(x+k) / (\$l-x * \$\mu-(x+k)))$  sums  $\$e \circ -x$   
if exponential-interpolation total-finite  $\bigwedge k::\text{nat}. \$\mu-(x+k) \neq 0$  for  $x::\text{nat}$   
*<proof>*

end

## 6.5 Limited Life Table

**locale** *limited-life-table* = *life-table* +  
assumes *l-limited:*  $\exists x::\text{real}. \$l-x = 0$   
**begin**

**lemma** *limited-survival-function-MM-X:* *limited-survival-function*  $\mathfrak{M} X$   
*<proof>*

end

**sublocale** *limited-life-table*  $\subseteq$  *limited-survival-function*  $\mathfrak{M} X$   
*<proof>*

**context** *limited-life-table*  
**begin**

**notation** *ult-age* ( $\langle \$\omega \rangle$ )

**lemma** *l-omega-0*:  $\$l-\$ \omega = 0$   
 $\langle proof \rangle$

**lemma** *l-0-equiv-nat*:  $\$l-x = 0 \iff x \geq \$\omega$  **for**  $x::nat$   
 $\langle proof \rangle$

**lemma** *d-l-equiv-nat*:  $\$d-\{t\&x\} = \$l-x \iff x+t \geq \$\omega$  **if**  $t \geq 0$  **for**  $x\ t :: nat$   
 $\langle proof \rangle$

**corollary** *d-1-omega-l*:  $\$d-(\$ \omega - 1) = \$l-(\$ \omega - 1)$   
 $\langle proof \rangle$

**lemma** *limited-life-table-imp-total-finite*: *total-finite*  
 $\langle proof \rangle$

**context**  
**fixes**  $x::nat$   
**assumes**  $x-lt-omega[simp]$ :  $x < \$\omega$   
**begin**

**lemma** *curt-e-sum-l-finite-nat*:  $\$e-x = (\sum_{k < n} \$l-(x+k+1)) / \$l-x$   
**if**  $\bigwedge k::nat. k < n \implies isCont\ l\ (x+k+1)\ x+n \geq \$\omega$  **for**  $n::nat$   
 $\langle proof \rangle$

**end**

**end**

**end**  
**theory** *Examples*  
**imports** *Life-Table*  
**begin**

## 7 Examples

The following lemma is a verification of the solution to the multiple choice question No. 3 of Exam LTAM Spring 2022 by Society of Actuaries.

**context** *smooth-survival-function*  
**begin**

**lemma** *SoA-LTAM-2022-Spring-MCQ-No3*:  
**assumes**  $\bigwedge x::real. 0 \leq x \implies x \leq 100 \implies cdf\ (distr\ \mathfrak{M}\ borel\ X)\ x = (1 - 0.01*x)^{\wedge}0.5$

**shows**  $|1000 * \mu - 25 - 6.7| < 0.05$   
 ⟨*proof*⟩

**end**

The following lemma is a verification of the solution to the problem No. 2. (1)-1 of Life Insurance Mathematics 2016 by the Institute of Actuaries of Japan, slightly modified; see the remark below.

**context** *smooth-life-table*

**begin**

**lemma** *IAJ-Life-Insurance-Math-2016-2-1-1*:

**fixes**  $a\ b :: \text{real}$

**assumes**  $-1 < a\ a < 0\ 0 < b - b/a \leq \psi$  **and**

*total-finite* **and**

$\bigwedge x. 0 < x \implies x < -b/a \implies l \text{ differentiable at } x$  **and**

$\bigwedge x. 0 \leq x \implies x < -b/a \implies e^{x \circ} = a * x + b$

**shows**  $\bigwedge x. 0 \leq x \implies x < -b/a \implies l x = l 0 * (b / (a * x + b))^{(a+1)/a}$

⟨*proof*⟩

REMARK. The original problem lacks the following hypotheses: (i)  $0 < b$ , (ii)  $-b/a \leq \psi$ , (iii)  $\forall x. 0 < x < -b/a \implies l \text{ differentiable at } x$ , (iv)  $\forall x. 0 \leq x < -b/a \implies l \text{ integrable-on } \{x..\}$ . Moreover, the hypothesis  $\forall x. 0 \leq x < -b/a$  is originally  $\forall x. 0 \leq x \leq -b/a$ .

**end**

**end**