

Abortable Linearizable Modules

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Abstract

We define the SLin I/O-automaton and prove its composition property. The SLin I/O-automaton is at the heart of the Speculative Linearizability framework. This framework simplifies devising robust distributed algorithms by enabling their decomposition into independent modules that can be analyzed and proved correct in isolation. It is particularly useful when working in a distributed environment, where the need to tolerate faults and asynchrony has made current monolithic protocols so intricate that it is no longer tractable to check their correctness. Our theory contains a formalization of simulation proof techniques in the I/O-automata of Lynch and Tuttle and a typical example of a refinement proof.

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1 Introduction

Linearizability [2] is a key design methodology for reasoning about implementations of concurrent abstract data types in both shared memory and message passing systems. It presents the illusion that operations execute sequentially and fault-free, despite the asynchrony and faults that are often present in a concurrent system, especially a distributed one.

However, devising complete linearizable objects is very difficult, especially in the presence of process crashes and asynchrony, requiring complex algorithms (such as Paxos [3]) to work correctly under general circumstances, and often resulting in bad average-case behavior. Concurrent algorithm designers therefore resort to speculation, i.e. to optimizing existing algorithms to handle common scenarios more efficiently. More precisely, a speculative systems has a fall-back mode that works in all situations and several optimization modes, each of which is very efficient in a particular situation but might not work at all in some other situation. By observing its execution, a speculative system speculates about which particular situation it will be subject to and chooses the most efficient mode for that situation. If speculation reveals wrong, a new speculation is made in light of newly available observations. Unfortunately, building speculative system ad-hoc results in protocols so complex that it is no longer tractable to prove their correctness.

We the specification of the SLin (a shorthand for Speculative Linearizability) I/O-automaton [5], which can be used to build a speculatively linearizable algorithm out of independent modules that each implement the different modes of the speculative algorithm. The SLin I/O-automaton is

at the heart of the Speculative Linearizability framework [4, 1]. The Speculative Linearizability framework first appeared in [1] and was later refined in [4]. This development is based on the later [4].

The SLin I/O-automaton produces traces that are linearizable with respect to a generic type of object. Moreover, the composition of two instances of the SLin I/O-automaton behaves like a single instance. Hence it is guaranteed that the composition of any number of instances of the SLin I/O-automaton is linearizable. In this formal development, we prove the idempotence theorem, i.e. that the composition of two instances of the SLin I/O-automaton is itself an implementation of the SLin I/O-automaton.

The properties stated above simplify the development and analysis of speculative systems: Instead of having to reason about an entanglement of complex protocols, one can devise several modules with the property that, when taken in isolation, each module refines the SLin I/O-automaton. Hence complex protocols can be divided into smaller modules that can be analyzed independently of each other. In particular, it allows to optimize an existing protocol by creating separate optimization modules, prove each optimization correct in isolation, and obtain the correctness of the overall protocol from the correctness of the existing one.

In this document we define the SLin I/O-automaton and prove the Composition Theorem, which states that the composition of two instances of the SLin I/O-automaton behaves as a single instance of the SLin I/O-automaton. We use a refinement mapping to establish this fact.

2 Sequences as Lists

```
theory Sequences
imports Main
begin
```

```
locale Sequences
begin
```

We reverse the order of application of (#) and (@) because it we think that it is easier to think of sequences as growing to the right.

```
no-notation Cons (infixr <#> 65)
abbreviation Append (infixl <#> 65)
  where Append xs x  $\equiv$  Cons x xs
no-notation append (infixr <@> 65)
abbreviation Concat (infixl <@> 65)
  where Concat xs ys  $\equiv$  append ys xs
```

```
end
```

```
end
```

3 I/O Automata with Finite-Trace Semantics

```
theory IOA
imports Main Sequences
begin
```

This theory is inspired and draws material from the IOA theory of Nipkow and Müller

```
locale IOA = Sequences
```

```
record 'a signature =
  inputs::'a set
  outputs::'a set
  internals::'a set
```

```
context IOA
begin
```

3.1 Signatures

```
definition actions :: 'a signature  $\Rightarrow$  'a set where
  actions asig  $\equiv$  inputs asig  $\cup$  outputs asig  $\cup$  internals asig
```

```
definition externals :: 'a signature  $\Rightarrow$  'a set where
  externals asig  $\equiv$  inputs asig  $\cup$  outputs asig
```

```
definition locals :: 'a signature  $\Rightarrow$  'a set where
  locals asig  $\equiv$  internals asig  $\cup$  outputs asig
```

```
definition is-asig :: 'a signature  $\Rightarrow$  bool where
  is-asig triple  $\equiv$ 
    inputs triple  $\cap$  outputs triple = {}  $\wedge$ 
    outputs triple  $\cap$  internals triple = {}  $\wedge$ 
    inputs triple  $\cap$  internals triple = {}
```

```
lemma internal-inter-external:
```

```
  assumes is-asig sig
  shows internals sig  $\cap$  externals sig = {}
  using assms by (auto simp add:internals-def externals-def is-asig-def)
```

```
definition hide-asig where
  hide-asig asig actns  $\equiv$ 
    (inputs = inputs asig - actns, outputs = outputs asig - actns,
     internals = internals asig  $\cup$  actns)
```

```
end
```

3.2 I/O Automata

```
type-synonym
```

(s, a) transition = $s \times a \times s$

record (s, a) ioa =
 asig :: a signature
 start :: s set
 trans :: (s, a) transition set

context IOA
begin

abbreviation $act A \equiv actions (asig A)$
abbreviation $ext A \equiv externals (asig A)$
abbreviation int **where** $int A \equiv internals (asig A)$
abbreviation $inp A \equiv inputs (asig A)$
abbreviation $out A \equiv outputs (asig A)$
abbreviation $local A \equiv locals (asig A)$

definition $is-ioa :: (s, a) ioa \Rightarrow bool$ **where**
 $is-ioa A \equiv is-asig (asig A)$
 $\wedge (\forall triple \in trans A . (fst \ o \ snd) \ triple \in act A)$

definition $hide$ **where**
 $hide A \ actns \equiv A(\lambda asig := hide-asig (asig A) \ actns)$

definition $is-trans :: s \Rightarrow a \Rightarrow (s, a) ioa \Rightarrow s \Rightarrow bool$ **where**
 $is-trans \ s1 \ a \ A \ s2 \equiv (s1, a, s2) \in trans A$

notation
 $is-trans \ (\leftarrow \ \text{-----} \ \rightarrow) [81, 81, 81, 81] \ 100$

definition $rename-set$ **where**
 $rename-set A \ ren \equiv \{b. \exists x \in A . ren \ b = Some \ x\}$

definition $rename$ **where**
 $rename A \ ren \equiv$
 $(\lambda asig = (\lambda inputs = rename-set (inp A) \ ren,$
 $outputs = rename-set (out A) \ ren,$
 $internals = rename-set (int A) \ ren),$
 $start = start A,$
 $trans = \{tr. \exists x . ren (fst (snd tr)) = Some \ x \wedge (fst \ tr) \ \text{-x-A} \longrightarrow (snd (snd$
 $tr))\})$

Reachable states and invariants

inductive
 $reachable :: (s, a) ioa \Rightarrow s \Rightarrow bool$
for $A :: (s, a) ioa$
where
 $reachable-0: s \in start A \implies reachable A \ s$
 $| \ reachable-n: [\ reachable A \ s; s \ \text{-a-A} \longrightarrow t] \implies reachable A \ t$

definition *invariant where*

invariant $A P \equiv (\forall s . \text{reachable } A s \longrightarrow P(s))$

theorem *invariantI:*

fixes $A P$

assumes $\bigwedge s . s \in \text{start } A \implies P s$

and $\bigwedge s t a . \llbracket \text{reachable } A s ; P s ; s -a-A \longrightarrow t \rrbracket \implies P t$

shows *invariant* $A P$

proof –

{ **fix** s

assume *reachable* $A s$

hence $P s$

proof (*induct rule:reachable.induct*)

fix s

assume $s \in \text{start } A$

thus $P s$ **using** *assms(1)* **by** *simp*

next

fix $a s t$

assume *reachable* $A s$ **and** $P s$ **and** $s -a-A \longrightarrow t$

thus $P t$ **using** *assms(2)* **by** *simp*

qed }

thus *?thesis* **by** (*simp add:invariant-def*)

qed

end

3.3 Composition of Families of I/O Automata

record (*'id, 'a*) *family* =

ids :: *'id set*

memb :: *'id* \Rightarrow *'a*

context *IOA*

begin

definition *is-ioa-fam where*

is-ioa-fam $\text{fam} \equiv \forall i \in \text{ids } \text{fam} . \text{is-ioa } (\text{memb } \text{fam } i)$

definition *compatible2 where*

compatible2 $A B \equiv$

out $A \cap \text{out } B = \{\}$ \wedge

int $A \cap \text{act } B = \{\}$ \wedge

int $B \cap \text{act } A = \{\}$

definition *compatible::('id, ('s,'a)ioa) family \Rightarrow bool where*

compatible $\text{fam} \equiv \text{finite } (\text{ids } \text{fam}) \wedge$

$(\forall i \in \text{ids } \text{fam} . \forall j \in \text{ids } \text{fam} . i \neq j \longrightarrow$

compatible2 (*memb* $\text{fam } i$) (*memb* $\text{fam } j$))

definition *asig-comp2* **where**

asig-comp2 $A B \equiv$
 $(inputs = (inputs A \cup inputs B) - (outputs A \cup outputs B),$
 $outputs = outputs A \cup outputs B,$
 $internals = internals A \cup internals B)$

definition *asig-comp*::('id, ('s,'a)ioa) family \Rightarrow 'a signature **where**

asig-comp fam \equiv
 $(inputs = \bigcup_{i \in (ids \text{ fam})}. inp (memb \text{ fam } i)$
 $- (\bigcup_{i \in (ids \text{ fam})}. out (memb \text{ fam } i)),$
 $outputs = \bigcup_{i \in (ids \text{ fam})}. out (memb \text{ fam } i),$
 $internals = \bigcup_{i \in (ids \text{ fam})}. int (memb \text{ fam } i))$

definition *par2* (infixr $\langle \parallel \rangle$ 10) **where**

$A \parallel B \equiv$
 $(asig = asig-comp2 (asig A) (asig B),$
 $start = \{pr. fst pr \in start A \wedge snd pr \in start B\},$
 $trans = \{tr.$
 $let s = fst tr; a = fst (snd tr); t = snd (snd tr)$
 $in (a \in act A \vee a \in act B)$
 $\wedge (if a \in act A$
 $then fst s -a-A \longrightarrow fst t$
 $else fst s = fst t)$
 $\wedge (if a \in act B$
 $then snd s -a-B \longrightarrow snd t$
 $else snd s = snd t) \}$)

definition *par*::('id, ('s,'a)ioa) family \Rightarrow ('id \Rightarrow 's,'a)ioa **where**

par fam \equiv let $ids = ids \text{ fam}; memb = memb \text{ fam}$ in
 $(asig = asig-comp \text{ fam},$
 $start = \{s . \forall i \in ids . s i \in start (memb i)\},$
 $trans = \{ (s, a, s') .$
 $(\exists i \in ids . a \in act (memb i))$
 $\wedge (\forall i \in ids .$
 $if a \in act (memb i)$
 $then s i -a-(memb i) \longrightarrow s' i$
 $else s i = (s' i) \})$

lemmas *asig-simps* = *hide-asig-def is-asig-def locals-def externals-def actions-def*
hide-def compatible-def asig-comp-def

lemmas *ioa-simps* = *rename-def rename-set-def is-trans-def is-ioa-def par-def*

end

3.4 Executions and Traces

type-synonym

('s,'a)pairs = ('a \times 's) list

type-synonym

$(s,a)execution = s \times (s,a)pairs$

type-synonym

$a trace = a list$

record $(s,a)execution-module =$

$execs::(s,a)execution set$

$asig::a signature$

record $a trace-module =$

$traces::a trace set$

$asig::a signature$

context IOA

begin

fun $is-exec-frag-of::(s,a)ioa \Rightarrow (s,a)execution \Rightarrow bool$ **where**

$is-exec-frag-of A (s,(ps\#p')\#p) =$

$(snd p' -fst p -A \longrightarrow snd p \wedge is-exec-frag-of A (s, (ps\#p')))$

$| is-exec-frag-of A (s, [p]) = s -fst p -A \longrightarrow snd p$

$| is-exec-frag-of A (s, []) = True$

definition $is-exec-of::(s,a)ioa \Rightarrow (s,a)execution \Rightarrow bool$ **where**

$is-exec-of A e \equiv fst e \in start A \wedge is-exec-frag-of A e$

definition $filter-act$ **where**

$filter-act \equiv map fst$

definition $schedule$ **where**

$schedule \equiv filter-act o snd$

definition $trace$ **where**

$trace sig \equiv filter (\lambda a . a \in externals sig) o schedule$

definition $is-schedule-of$ **where**

$is-schedule-of A sch \equiv$

$(\exists e . is-exec-of A e \wedge sch = filter-act (snd e))$

definition $is-trace-of$ **where**

$is-trace-of A tr \equiv$

$(\exists sch . is-schedule-of A sch \wedge tr = filter (\lambda a . a \in ext A) sch)$

definition $traces$ **where**

$traces A \equiv \{tr . is-trace-of A tr\}$

lemma $traces-alt:$

shows $traces A = \{tr . \exists e . is-exec-of A e$

$\wedge tr = trace (ioa.asig A) e\}$

proof –


```

{ fix t
  assume a:t ∈ traces A
  have ∃ e . is-exec-of A e ∧ trace (ioa.asig A) e = t
  proof –
    from a obtain sch where 1:is-schedule-of A sch
      and 2:t = filter (λ a. a ∈ ext A) sch
      by (auto simp add:traces-def is-trace-of-def)
    from 1 obtain e where 3:is-exec-of A e and 4:sch = filter-act (snd e)
      by (auto simp add:is-schedule-of-def)
    from 4 and 2 have trace (ioa.asig A) e = t
      by (simp add:trace-def schedule-def)
    with 3 show ?thesis by fast
  qed }
moreover
{ fix e
  assume is-exec-of A e
  hence trace (ioa.asig A) e ∈ traces A
    by (force simp add:trace-def schedule-def traces-def
      is-trace-of-def is-schedule-of-def is-exec-of-def) }
ultimately show ?thesis by blast
qed

```

lemmas *trace-simps* = *traces-def is-trace-of-def is-schedule-of-def filter-act-def is-exec-of-def trace-def schedule-def*

definition *proj-trace*::'a trace ⇒ ('a signature) ⇒ 'a trace (**infixr** <|> 12) **where**
proj-trace t sig ≡ *filter (λ a . a ∈ actions sig) t*

definition *ioa-implements* :: ('s1,'a)ioa ⇒ ('s2,'a)ioa ⇒ bool (**infixr** <=<|> 12)
where
A =<| B ≡ *inp A = inp B ∧ out A = out B ∧ traces A ⊆ traces B*

3.5 Operations on Executions

definition *cons-exec* **where**
cons-exec e p ≡ *(fst e, (snd e)#p)*

definition *append-exec* **where**
append-exec e e' ≡ *(fst e, (snd e)@(snd e'))*

fun *last-state* **where**
last-state (s,[]) = *s*
| *last-state (s,ps#p)* = *snd p*

lemma *last-state-reachable*:
fixes *A e*
assumes *is-exec-of A e*
shows *reachable A (last-state e)* **using** *assms*
proof –

have *is-exec-of* A $e \implies$ *reachable* A (*last-state* e)
proof (*induction snd e arbitrary: e*)
 case *Nil*
 from *Nil.prem*s **have** $1:fst\ e \in start\ A$ **by** (*simp add:is-exec-of-def*)
 from *Nil.hyps* **have** $2:last-state\ e = fst\ e$ **by** (*metis last-state.simps(1) surjective-pairing*)
 from 1 **and** 2 **and** *Nil.hyps* **show** *?case* **by** (*metis reachable-0*)
next
 case (*Cons p ps e*)
 let $?e' = (fst\ e, ps)$
 have *ih:reachable* A (*last-state* $?e'$)
 proof –
 from *Cons.prem*s **and** *Cons.hyps(2)* **have** *is-exec-of* A $?e'$
 by (*simp add:is-exec-of-def*)
 (*metis (full-types) IOA.is-exec-frag-of.simps(1) IOA.is-exec-frag-of.simps(3)*
 neq-Nil-conv prod.collapse)
 with *Cons.hyps(1)* **show** *?thesis* **by** *auto*
 qed
 from *Cons.prem*s **and** *Cons.hyps(2)* **have** (*last-state* $?e'$)–(*fst p*)– $A \longrightarrow$ (*snd p*)
 by (*simp add:is-exec-of-def*) (*cases (A, fst e, ps#p) rule:is-exec-frag-of.cases, auto*)
 with *ih* **and** *Cons.hyps(2)* **show** *?case*
 by (*metis last-state.simps(2) reachable.simps surjective-pairing*)
 qed
 thus *?thesis* **using** *assms* **by** *fastforce*
qed

lemma *trans-from-last-state*:

assumes *is-exec-frag-of* A e **and** (*last-state* e)– a – $A \longrightarrow s'$
shows *is-exec-frag-of* A (*cons-exec* e (a, s'))
using *assms* **by** (*cases (A, fst e, snd e) rule:is-exec-frag-of.cases, auto simp add:cons-exec-def*)

lemma *exec-frag-prefix*:

fixes $A\ p\ ps$
assumes *is-exec-frag-of* A (*cons-exec* $e\ p$)
shows *is-exec-frag-of* $A\ e$
using *assms* **by** (*cases (A, fst e, snd e) rule:is-exec-frag-of.cases, auto simp add:cons-exec-def*)

lemma *trace-same-ext*:

fixes $A\ B\ e$
assumes *ext* $A = ext\ B$
shows *trace* (*ioa.asig* A) $e = trace$ (*ioa.asig* B) e
using *assms* **by** (*auto simp add:trace-def*)

lemma *trace-append-is-append-trace*:

```

fixes e e' sig
shows trace sig (append-exec e' e) = trace sig e' @ trace sig e
by (simp add:append-exec-def trace-def schedule-def filter-act-def)

lemma append-exec-frags-is-exec-frag:
  fixes e e' A as
  assumes is-exec-frag-of A e and last-state e = fst e'
  and is-exec-frag-of A e'
  shows is-exec-frag-of A (append-exec e e')
proof –
  from assms show ?thesis
  proof (induct (fst e',snd e') arbitrary:e' rule:is-exec-frag-of.induct)
    case (3 A)
    from 3.hyps and 3.prem1
    show ?case by (simp add:append-exec-def)
  next
    case (2 A p)
    have last-state e –(fst p)–A→ snd p using 2.prem2,3 and 2.hyps
      by (metis is-exec-frag-of.simp2 prod.collapse)
    hence is-exec-frag-of A (fst e, (snd e)#p) using 2.prem1
      by (metis cons-exec-def prod.collapse trans-from-last-state)
    moreover
    have append-exec e e' = (fst e, (snd e)#p) using 2.hyps
      by (metis append-Cons append-Nil append-exec-def)
    ultimately
    show ?case by auto
  next
    case (1 A ps p' p e')
    have is-exec-frag-of A (fst e, (snd e)@((ps#p')#p))
    proof –
      have is-exec-frag-of A (fst e, (snd e)@(ps#p'))
        by (metis 1.hyps 1.prem1 append-exec-def cons-exec-def
          exec-frag-prefix fst-conv prod-eqI snd-conv)
      moreover
      have snd p' –(fst p)–A→ snd p using 1.prem3 1.hyps2
        by (metis is-exec-frag-of.simp1 prod.collapse)
      ultimately show ?thesis by simp
    qed
    moreover have append-exec e e' = (fst e, (snd e)@((ps#p')#p))
      by (metis 1.hyps2 append-exec-def)
    ultimately show ?case by simp
  qed
qed

lemma last-state-of-append:
  fixes e e'
  assumes fst e' = last-state e
  shows last-state (append-exec e e') = last-state e'
  using assms by (cases e' rule:last-state.cases, auto simp add:append-exec-def)

```

end

end

4 Recoverable Data Types

theory *RDR*
imports *Main Sequences*
begin

4.1 The pre-RDR locale contains definitions later used in the RDR locale to state the properties of RDRs

locale *pre-RDR* = *Sequences* +
fixes $\delta::'a \Rightarrow ('b \times 'c) \Rightarrow 'a$ (infix $\langle \cdot \rangle$ 65)
and $\gamma::'a \Rightarrow ('b \times 'c) \Rightarrow 'd$
and *bot*:: $'a (\langle \perp \rangle)$
begin

fun *exec*:: $'a \Rightarrow ('b \times 'c)list \Rightarrow 'a$ (infix $\langle \star \rangle$ 65) where
exec *s Nil* = *s*
| *exec* *s (rs#r)* = (*exec* *s rs*) \cdot *r*

definition *less-eq* (infix $\langle \preceq \rangle$ 50) where
less-eq *s s'* $\equiv \exists rs . s' = (s \star rs)$

definition *less* (infix $\langle \prec \rangle$ 50) where
less *s s'* \equiv *less-eq* *s s'* $\wedge s \neq s'$

definition *is-lb* where
is-lb *s s1 s2* $\equiv s \preceq s2 \wedge s \preceq s1$

definition *is-glb* where
is-glb *s s1 s2* \equiv *is-lb* *s s1 s2* $\wedge (\forall s' . \text{is-lb } s' s1 s2 \longrightarrow s' \preceq s)$

definition *contains* where
contains *s r* $\equiv \exists rs . r \in \text{set } rs \wedge s = (\perp \star rs)$

definition *inf* (infix $\langle \sqcap \rangle$ 65) where
inf *s1 s2* \equiv *THE* *s . is-glb* *s s1 s2*

4.2 Useful Lemmas in the pre-RDR locale

lemma *exec-cons*:
 $s \star (rs \# r) = (s \star rs) \cdot r$ by *simp*

lemma *exec-append*:
 $(s \star rs) \star rs' = s \star (rs @ rs')$

proof (*induct rs*[^])
show $(s \star rs) \star [] = s \star (rs @ [])$ **by** *simp*
next
fix $rs' r$
assume $ih:(s \star rs) \star rs' = s \star (rs @ rs')$
thus $(s \star rs) \star (rs' \# r) = s \star (rs @ (rs' \# r))$
by (*metis append-Cons exec-cons*)
qed

lemma *trans*:
assumes $s1 \preceq s2$ **and** $s2 \preceq s3$
shows $s1 \preceq s3$ **using** *assms*
by (*auto simp add:less-eq-def, metis exec-append*)

lemma *contains-star*:
fixes $s r rs$
assumes *contains s r*
shows *contains (s \star rs) r*
proof (*induct rs*)
case *Nil*
show *contains (s \star []) r* **using** *assms* **by** *auto*
next
case (*Cons r' rs*)
with this **obtain** rs' **where** $1:s \star rs = \perp \star rs'$ **and** $2:r \in \text{set } rs'$
by (*auto simp add:contains-def*)
have $3:s \star (rs \# r') = \perp \star (rs' \# r')$ **using** 1 **by** *fastforce*
show *contains (s \star (rs \# r')) r* **using** $2\ 3$
by (*auto simp add:contains-def*) (*metis exec-cons rev-subsetD set-subset-Cons*)
qed

lemma *preceq-star*: $s \star (rs \# r) \preceq s' \implies s \star rs \preceq s'$
by (*metis pre-RDR.exec.simps(1) pre-RDR.exec.simps(2) pre-RDR.less-eq-def trans*)

end

4.3 The RDR locale

locale *RDR* = *pre-RDR* +
assumes *idem1:contains s r* $\implies s \cdot r = s$
and *idem2*: $\bigwedge s r r' . \text{fst } r \neq \text{fst } r' \implies \gamma s r = \gamma ((s \cdot r) \cdot r')$ *r*
and *antisym*: $\bigwedge s1 s2 . s1 \preceq s2 \wedge s2 \preceq s1 \implies s1 = s2$
and *glb-exists*: $\bigwedge s1 s2 . \exists s . \text{is-glb } s s1 s2$
and *consistency*: $\bigwedge s1 s2 s3 rs . s1 \preceq s2 \implies s2 \preceq s3 \implies s3 = s1 \star rs$
 $\implies \exists rs' rs'' . s2 = s1 \star rs' \wedge s3 = s2 \star rs''$
 $\wedge \text{set } rs' \subseteq \text{set } rs \wedge \text{set } rs'' \subseteq \text{set } rs$
and *bot*: $\bigwedge s . \perp \preceq s$
begin

lemma *inf-glb:is-glb (s1 \sqcap s2) s1 s2*

proof –
 { **fix** $s s'$
 assume $is\text{-glb } s s1 s2$ **and** $is\text{-glb } s' s1 s2$
 hence $s = s'$ **using** $antisym$ **by** $(auto simp add:is\text{-glb-def } is\text{-lb-def})$ }
from this and $glb\text{-exists}$ **show** $?thesis$
 by $(auto simp add:inf\text{-def, metis (lifting) theI'})$
qed

sublocale $ordering\ less\text{-eq } less$

proof
fix s
show $s \preceq s$
by $(metis exec.simps(1) less\text{-eq-def})$
next
fix $s s'$
show $s < s' = (s \preceq s' \wedge s \neq s')$
by $(auto simp add:less\text{-def})$
next
fix $s s'$
assume $s \preceq s'$ **and** $s' \preceq s$
thus $s = s'$
using $antisym$ **by** $auto$
next
fix $s1 s2 s3$
assume $s1 \preceq s2$ **and** $s2 \preceq s3$
thus $s1 \preceq s3$
using $trans$ **by** $blast$
qed

sublocale $semilattice\text{-set } inf$

proof
fix s
show $s \sqcap s = s$
 using $inf\text{-gll}$
 by $(metis antisym is\text{-glb-def } is\text{-lb-def } refl)$
next
fix $s1 s2$
show $s1 \sqcap s2 = (s2 \sqcap s1)$
 using $inf\text{-gll}$
 by $(smt antisym is\text{-glb-def } pre\text{-RDR.is-lb-def})$
next
fix $s1 s2 s3$
show $(s1 \sqcap s2) \sqcap s3 = (s1 \sqcap (s2 \sqcap s3))$
 using $inf\text{-gll}$
 by $(auto simp add:is\text{-glb-def } is\text{-lb-def, smt antisym trans})$
qed

sublocale $semilattice\text{-order-set } inf\ less\text{-eq } less$

proof

```

fix s s'
show s  $\preceq$  s' = (s = s  $\sqcap$  s')
by (metis antisym idem inf-glb pre-RDR.is-glb-def pre-RDR.is-lb-def)
next
fix s s'
show s  $\prec$  s' = (s = s  $\sqcap$  s'  $\wedge$  s  $\neq$  s')
by (metis inf-glb local.antisym local.refl pre-RDR.is-glb-def pre-RDR.is-lb-def
pre-RDR.less-def)
qed

```

notation $F (\langle \sqcap \rightarrow [99]$)

4.4 Some useful lemmas

lemma *idem-star*:

```

fixes r s rs
assumes contains s r
shows s  $\star$  rs = s  $\star$  (filter ( $\lambda x . x \neq r$ ) rs)
proof (induct rs)
  case Nil
    show s  $\star$  [] = s  $\star$  (filter ( $\lambda x . x \neq r$ ) [])
      using assms by auto
  next
    case (Cons r' rs)
      have 1:contains (s  $\star$  rs) r using assms and contains-star by auto
      show s  $\star$  (rs#r') = s  $\star$  (filter ( $\lambda x . x \neq r$ ) (rs#r'))
      proof (cases r' = r)
        case True
          hence s  $\star$  (rs#r') = s  $\star$  rs using idem1 1 by auto
          thus ?thesis using Cons by simp
        case False
          thus ?thesis using Cons by auto
      qed
  qed

```

lemma *idem-star2*:

```

fixes s rs'
shows  $\exists$  rs' . s  $\star$  rs = s  $\star$  rs'  $\wedge$  set rs'  $\subseteq$  set rs
   $\wedge$  ( $\forall r \in$  set rs' .  $\neg$  contains s r)
proof (induct rs)
  case Nil
    thus  $\exists$  rs' . s  $\star$  [] = s  $\star$  rs'  $\wedge$  set rs'  $\subseteq$  set []
       $\wedge$  ( $\forall r \in$  set rs' .  $\neg$  contains s r) by force
  next
    case (Cons r rs)
      obtain rs' where 1:s  $\star$  rs = s  $\star$  rs' and 2:set rs'  $\subseteq$  set rs
      and 3: $\forall r \in$  set rs' .  $\neg$  contains s r using Cons(1) by blast
      show  $\exists$  rs' . s  $\star$  (rs#r) = s  $\star$  rs'  $\wedge$  set rs'  $\subseteq$  set (rs#r)

```

$\wedge (\forall r \in \text{set } rs' . \neg \text{contains } s r)$
proof (*cases contains s r*)
case *True*
have $s \star (rs \# r) = s \star rs'$
proof –
have $s \star (rs \# r) = s \star rs$ **using** *True*
by (*metis contains-star exec-cons idem1*)
moreover
have $s \star (rs' \# r) = s \star rs'$ **using** *True*
by (*metis contains-star exec-cons idem1*)
ultimately show *?thesis* **using** *1* **by** *simp*
qed
moreover **have** $\text{set } rs' \subseteq \text{set } (rs \# r)$ **using** *2*
by (*simp, metis subset-insertI2*)
moreover **have** $\forall r \in \text{set } rs' . \neg \text{contains } s r$
using *3* **by** *assumption*
ultimately show *?thesis* **by** *blast*
next
case *False*
have $s \star (rs \# r) = s \star (rs' \# r)$ **using** *1* **by** *simp*
moreover
have $\text{set } (rs' \# r) \subseteq \text{set } (rs \# r)$ **using** *2* **by** *auto*
moreover **have** $\forall r \in \text{set } (rs' \# r) . \neg \text{contains } s r$
using *3* **by** *False by auto*
ultimately show *?thesis* **by** *blast*
qed
qed

lemma *idem2-star*:
assumes *contains s r*
and $\bigwedge r' . r' \in \text{set } rs \implies \text{fst } r' \neq \text{fst } r$
shows $\gamma s r = \gamma (s \star rs) r$ **using** *assms*
proof (*induct rs*)
case *Nil*
show $\gamma s r = \gamma (s \star []) r$ **by** *simp*
next
case (*Cons r' rs*)
thus $\gamma s r = \gamma (s \star (rs \# r')) r$
using *assms* **by** *auto*
(metis contains-star fst-conv idem1 idem2 prod.exhaust)
qed

lemma *glb-common*:
fixes $s1 s2 s rs1 rs2$
assumes $s1 = s \star rs1$ **and** $s2 = s \star rs2$
shows $\exists rs . s1 \sqcap s2 = s \star rs \wedge \text{set } rs \subseteq \text{set } rs1 \cup \text{set } rs2$
proof –
have $1:s \preceq s1$ **and** $2:s \preceq s2$ **using** *assms* **by** (*auto simp add:less-eq-def*)
hence $3:s \preceq s1 \sqcap s2$ **by** (*metis inf-glb is-lb-def pre-RDR.is-glb-def*)


```

  have  $\_4:s1 \sqcap s2 \preceq s1$  by (metis cobounded1)
  show ?thesis using  $\_3 \_4$  assms(1) and consistency by blast
qed

```

lemma *glb-common-set*:

```
fixes ss s0 rset
```

```
assumes finite ss and  $ss \neq \{\}$ 
```

```
and  $\bigwedge s . s \in ss \implies \exists rs . s = s0 \star rs \wedge set\ rs \subseteq rset$ 
```

```
shows  $\exists rs . \bigsqcap ss = s0 \star rs \wedge set\ rs \subseteq rset$ 
```

```
using assms
```

```
proof (induct ss rule:finite-ne-induct)
```

```
  case (singleton s)
```

```
  obtain rs where  $s = s0 \star rs \wedge set\ rs \subseteq rset$  using singleton by force
```

```
  moreover have  $\bigsqcap \{s\} = s$  using singleton by auto
```

```
  ultimately show  $\exists rs . \bigsqcap \{s\} = s0 \star rs \wedge set\ rs \subseteq rset$  by blast
```

```
next
```

```
  case (insert s ss)
```

```
  have  $1:\bigwedge s' . s' \in ss \implies \exists rs . s' = s0 \star rs \wedge set\ rs \subseteq rset$ 
```

```
    using insert(5) by force
```

```
  obtain rs where  $2:\bigsqcap ss = s0 \star rs$  and  $3:set\ rs \subseteq rset$ 
```

```
    using insert(4) 1 by blast
```

```
  obtain rs' where  $4:s = s0 \star rs'$  and  $5:set\ rs' \subseteq rset$ 
```

```
    using insert(5) by blast
```

```
  have  $6:\bigsqcap (insert\ s\ ss) = s \sqcap (\bigsqcap ss)$ 
```

```
    by (metis insert.hyps(1-3) insert-not-elem)
```

```
  obtain rs'' where  $7:\bigsqcap (insert\ s\ ss) = s0 \star rs''$ 
```

```
    and  $8:set\ rs'' \subseteq set\ rs' \cup set\ rs$ 
```

```
    using glb-common 2 4 6 by force
```

```
  have  $9:set\ rs'' \subseteq rset$  using 3 5 8 by blast
```

```
  show  $\exists rs . \bigsqcap (insert\ s\ ss) = s0 \star rs \wedge set\ rs \subseteq rset$ 
```

```
    using 7 9 by blast
```

```
qed
```

```
end
```

```
end
```

5 The SLin Automata specification

```
theory SLin
```

```
imports IOA RDR
```

```
begin
```

```
datatype ('a,'b,'c,'d)SLin-action =
```

```
— The nat component is the instance number
```

```
  Invoke nat 'b 'c
```

```
| Response nat 'b 'd
```

```
| Switch nat 'b 'c 'a
```

```
| Recover nat
```

| *Linearize nat*

datatype *SLin-status* = *Sleep* | *Pending* | *Ready* | *Aborted*

record (*'a, 'b, 'c*)*SLin-state* =
pending :: *'b* ⇒ *'b* × *'c*
initVals :: *'a* set
abortVals :: *'a* set
status :: *'b* ⇒ *SLin-status*
dstate :: *'a*
initialized :: *bool*

locale *SLin* = *RDR* + *IOA*

begin

definition

asig :: *nat* ⇒ *nat* ⇒ (*'a, 'b, 'c, 'd*)*SLin-action* signature

— The first instance has number 0

where

asig *i j* ≡ ()
inputs = {*act* . ∃ *p c* *iw i'* .
(*i* ≤ *i'* ∧ *i'* < *j* ∧ *act* = *Invoke i' p c*) ∨ (*i* > 0 ∧ *act* = *Switch i p c iw*)},
outputs = {*act* . ∃ *p c* *av i' outp* .
(*i* ≤ *i'* ∧ *i'* < *j* ∧ *act* = *Response i' p outp*) ∨ *act* = *Switch j p c av*},
internals = {*act*. ∃ *i'* . *i* ≤ *i'* ∧ *i'* < *j*
∧ (*act* = *Linearize i'* ∨ *act* = *Recover i'*)} ()

definition *pendingReqs* :: (*'a, 'b, 'c*)*SLin-state* ⇒ (*'b* × *'c*) set

where

pendingReqs *s* ≡ {*r* . ∃ *p* .
r = *pending* *s* *p*
∧ *status* *s* *p* ∈ {*Pending*, *Aborted*}}

definition *Inv* :: *'b* ⇒ *'c*

⇒ (*'a, 'b, 'c*)*SLin-state* ⇒ (*'a, 'b, 'c*)*SLin-state* ⇒ *bool*

where

Inv *p c s s'* ≡
status *s* *p* = *Ready*
∧ *s'* = *s*(*pending* := (*pending* *s*)(*p* := (*p, c*)),
status := (*status* *s*)(*p* := *Pending*))

definition *pendingSeqs* **where**

pendingSeqs *s* ≡ {*rs* . set *rs* ⊆ *pendingReqs* *s*}

definition *Lin* :: (*'a, 'b, 'c*)*SLin-state* ⇒ (*'a, 'b, 'c*)*SLin-state* ⇒ *bool*

where

Lin *s s'* ≡ ∃ *rs* ∈ *pendingSeqs* *s* .
initialized *s*
∧ (∀ *av* ∈ *abortVals* *s* . (*dstate* *s*) ★ *rs* ≼ *av*)

$$\wedge s' = s(\text{dstate} := (\text{dstate } s) \star rs)$$

definition *initSets* **where**

$$\text{initSets } s \equiv \{ivs . ivs \neq \{\}\} \wedge ivs \subseteq \text{initVals } s\}$$

definition *safeInits* **where**

$$\begin{aligned} \text{safeInits } s &\equiv \text{if } \text{initVals } s = \{\} \text{ then } \{\} \\ &\text{else } \{d . \exists ivs \in \text{initSets } s . \exists rs \in \text{pendingSeqs } s . \\ &\quad d = \prod ivs \star rs \wedge (\forall av \in \text{abortVals } s . d \preceq av)\} \end{aligned}$$

definition *initAborts* **where**

$$\begin{aligned} \text{initAborts } s &\equiv \{d . \text{dstate } s \preceq d \\ &\wedge ((\exists rs \in \text{pendingSeqs } s . d = \text{dstate } s \star rs) \\ &\vee (\exists ivs \in \text{initSets } s . \text{dstate } s \preceq \prod ivs \\ &\wedge (\exists rs \in \text{pendingSeqs } s . d = \prod ivs \star rs)))\} \end{aligned}$$

definition *uninitAborts* **where**

$$\begin{aligned} \text{uninitAborts } s &\equiv \{d . \\ &\exists ivs \in \text{initSets } s . \exists rs \in \text{pendingSeqs } s . \\ &\quad d = \prod ivs \star rs\} \end{aligned}$$

definition *safeAborts*::('a,'b,'c)SLin-state \Rightarrow 'a set **where**

$$\begin{aligned} \text{safeAborts } s &\equiv \text{if } \text{initialized } s \text{ then } \text{initAborts } s \\ &\text{else } \text{uninitAborts } s \end{aligned}$$

definition *Reco* :: ('a,'b,'c)SLin-state \Rightarrow ('a,'b,'c)SLin-state \Rightarrow bool **where**

$$\begin{aligned} \text{Reco } s \ s' &\equiv \\ &(\exists p . \text{status } s \ p \neq \text{Sleep}) \\ &\wedge \neg \text{initialized } s \\ &\wedge (\exists d \in \text{safeInits } s . \\ &\quad s' = s(\text{dstate} := d, \text{initialized} := \text{True})) \end{aligned}$$

definition *Resp* :: 'b \Rightarrow 'd \Rightarrow ('a,'b,'c)SLin-state \Rightarrow ('a,'b,'c)SLin-state \Rightarrow bool **where**

$$\begin{aligned} \text{Resp } p \text{ ou } s \ s' &\equiv \\ &\text{status } s \ p = \text{Pending} \\ &\wedge \text{initialized } s \\ &\wedge \text{contains } (\text{dstate } s) (\text{pending } s \ p) \\ &\wedge \text{ou} = \gamma (\text{dstate } s) (\text{pending } s \ p) \\ &\wedge s' = s(\text{status} := (\text{status } s)(p := \text{Ready})) \end{aligned}$$

definition *Init* :: 'b \Rightarrow 'c \Rightarrow 'a

$$\Rightarrow ('a,'b,'c)SLin-state \Rightarrow ('a,'b,'c)SLin-state \Rightarrow \text{bool}$$

where

$$\begin{aligned} \text{Init } p \ c \ iv \ s \ s' &\equiv \\ &\text{status } s \ p = \text{Sleep} \\ &\wedge s' = s(\text{initVals} := \{iv\} \cup (\text{initVals } s), \\ &\quad \text{status} := (\text{status } s)(p := \text{Pending}), \\ &\quad \text{pending} := (\text{pending } s)(p := (p,c))) \end{aligned}$$

definition *Abort* :: 'b ⇒ 'c ⇒ 'a
 ⇒ ('a,'b,'c)SLin-state ⇒ ('a,'b,'c)SLin-state ⇒ bool
where
Abort p c av s s' ≡
 status s p = Pending ∧ pending s p = (p,c)
 ∧ av ∈ safeAborts s
 ∧ s' = s(status := (status s)(p := Aborted),
 abortVals := (abortVals s ∪ {av}))

definition *trans* **where**
trans i j ≡ { (s,a,s') . case a of
 | Invoke i' p c ⇒ i ≤ i' ∧ i < j ∧ Inv p c s s'
 | Response i' p ou ⇒ i ≤ i' ∧ i < j ∧ Resp p ou s s'
 | Switch i' p c v ⇒ (i > 0 ∧ i' = i ∧ Init p c v s s')
 ∨ (i' = j ∧ Abort p c v s s')
 | Linearize i' ⇒ i' = i ∧ Lin s s'
 | Recover i' ⇒ i > 0 ∧ i' = i ∧ Reco s s' }

definition *start* **where**
start i ≡ { s .
 ∀ p . status s p = (if i > 0 then Sleep else Ready)
 ∧ dstate s = ⊥
 ∧ (if i > 0 then ¬ initialized s else initialized s)
 ∧ initVals s = {}
 ∧ abortVals s = {} }

definition *ioa* **where**
ioa i j ≡
 (ioa.asig = asig i j ,
 start = start i,
 trans = trans i j)

end

end

6 Definition and Soundness of Refinement Mappings, Forward Simulations and Backward Simulations

theory *Simulations*
imports *IOA*
begin

context *IOA*
begin

definition *refines where*

refines e s a t A f \equiv *fst e = f s* \wedge *last-state e = f t* \wedge *is-exec-frag-of A e*
 \wedge (*let tr = trace (ioa.asig A) e in*
if a \in *ext A* *then tr = [a]* *else tr = []*)

definition

is-ref-map $::$ (*'s1* \Rightarrow *'s2*) \Rightarrow (*'s1, 'a*)*ioa* \Rightarrow (*'s2, 'a*)*ioa* \Rightarrow *bool* **where**
is-ref-map f B A \equiv
 $(\forall s \in \text{start } B . f s \in \text{start } A) \wedge (\forall s t a . \text{reachable } B s \wedge s -a-B \longrightarrow t$
 $\longrightarrow (\exists e . \text{refines } e s a t A f))$

definition

is-forward-sim $::$ (*'s1* \Rightarrow (*'s2 set*)) \Rightarrow (*'s1, 'a*)*ioa* \Rightarrow (*'s2, 'a*)*ioa* \Rightarrow *bool* **where**
is-forward-sim f B A \equiv
 $(\forall s \in \text{start } B . f s \cap \text{start } A \neq \{\})$
 $\wedge (\forall s s' t a . s' \in f s \wedge s -a-B \longrightarrow t \wedge \text{reachable } B s$
 $\longrightarrow (\exists e . \text{fst } e = s' \wedge \text{last-state } e \in f t \wedge \text{is-exec-frag-of } A e$
 $\wedge (\text{let } tr = \text{trace } (ioa.asig A) e \text{ in}$
 $\text{if } a \in \text{ext } A \text{ then } tr = [a] \text{ else } tr = []))$

definition

is-backward-sim $::$ (*'s1* \Rightarrow (*'s2 set*)) \Rightarrow (*'s1, 'a*)*ioa* \Rightarrow (*'s2, 'a*)*ioa* \Rightarrow *bool* **where**
is-backward-sim f B A \equiv
 $(\forall s . f s \neq \{\})$ — Quantifying over reachable states would suffice
 $\wedge (\forall s \in \text{start } B . f s \subseteq \text{start } A)$
 $\wedge (\forall s t a t' . t' \in f t \wedge s -a-B \longrightarrow t \wedge \text{reachable } B s$
 $\longrightarrow (\exists e . \text{fst } e \in f s \wedge \text{last-state } e = t' \wedge \text{is-exec-frag-of } A e$
 $\wedge (\text{let } tr = \text{trace } (ioa.asig A) e \text{ in}$
 $\text{if } a \in \text{ext } A \text{ then } tr = [a] \text{ else } tr = []))$

6.1 A series of lemmas that will be useful in the soundness proofs

lemma *step-eq-traces:*

fixes *e-B' A e e-A' a t*

defines *e-A* \equiv *append-exec e-A' e* **and** *e-B* \equiv *cons-exec e-B' (a, t)*

and *tr* \equiv *trace (ioa.asig A) e*

assumes *1:trace (ioa.asig A) e-A' = trace (ioa.asig A) e-B'*

and *2:if a* \in *ext A* *then tr = [a]* *else tr = []*

shows *trace (ioa.asig A) e-A = trace (ioa.asig A) e-B*

proof —

have *3:trace (ioa.asig A) e-B =*

(if a \in *ext A* *then (trace (ioa.asig A) e-B') # a* *else trace (ioa.asig A) e-B')*

using *e-B-def* **by** (*simp add:trace-def schedule-def filter-act-def cons-exec-def*)

have *4:trace (ioa.asig A) e-A =*

(if a \in *ext A* *then trace (ioa.asig A) e-A' # a* *else trace (ioa.asig A) e-A')*

using *2 trace-append-is-append-trace*[*of ioa.asig A e-A' e*]

by(*auto simp add:e-A-def tr-def split: if-split-asm*)

show *?thesis* **using** *1 3 4* **by** *simp*

qed

lemma *exec-inc-imp-trace-inc*:
fixes $A B$
assumes $\text{ext } B = \text{ext } A$
and $\bigwedge e-B . \text{is-exec-of } B e-B$
 $\implies \exists e-A . \text{is-exec-of } A e-A \wedge \text{trace } (\text{ioa.asig } A) e-A = \text{trace } (\text{ioa.asig } A) e-B$
shows $\text{traces } B \subseteq \text{traces } A$
proof –
{ **fix** t
assume $t \in \text{traces } B$
with this obtain e **where** $1:t = \text{trace } (\text{ioa.asig } B) e$ **and** $2:\text{is-exec-of } B e$
using *traces-alt* *assms(1)* **by** *blast*
from 1 **and** *assms(1)* **have** $3:t = \text{trace } (\text{ioa.asig } A) e$ **by** (*simp add:trace-def*)
from 2 3 **and** *assms(2)* **obtain** e' **where**
 $\text{is-exec-of } A e' \wedge \text{trace } (\text{ioa.asig } A) e' = \text{trace } (\text{ioa.asig } A) e$ **by** *blast*
hence $t \in \text{traces } A$ **using** 3 *traces-alt* **by** *fastforce* }
thus *?thesis* **by** *fast*
qed

6.2 Soundness of Refinement Mappings

lemma *ref-map-execs*:
fixes $A::('sA, 'a)\text{ioa}$ **and** $B::('sB, 'a)\text{ioa}$ **and** $f::'sB \Rightarrow 'sA$ **and** $e-B$
assumes *is-ref-map* $f B A$ **and** *is-exec-of* $B e-B$
shows $\exists e-A . \text{is-exec-of } A e-A$
 $\wedge \text{trace } (\text{ioa.asig } A) e-A = \text{trace } (\text{ioa.asig } A) e-B$
proof –
note *assms(2)*
hence $\exists e-A . \text{is-exec-of } A e-A$
 $\wedge \text{trace } (\text{ioa.asig } A) e-A = \text{trace } (\text{ioa.asig } A) e-B$
 $\wedge \text{last-state } e-A = f (\text{last-state } e-B)$
proof (*induction snd e-B arbitrary:e-B*)
case *Nil*
let $?e-A = (f (\text{fst } e-B), [])$
have $\bigwedge s . s \in \text{start } B \implies f s \in \text{start } A$ **using** *assms(1)* **by** (*simp add:is-ref-map-def*)
hence *is-exec-of* $A ?e-A$ **using** *Nil.premis(1)* **by** (*simp add:is-exec-of-def*)
moreover
have $\text{trace } (\text{ioa.asig } A) ?e-A = \text{trace } (\text{ioa.asig } A) e-B$
using *Nil.hyps* **by** (*simp add:trace-simps*)
moreover
have $\text{last-state } ?e-A = f (\text{last-state } e-B)$
using *Nil.hyps* **by** (*metis last-state.simps(1) prod.collapse*)
ultimately show *?case* **by** *fast*
next
case (*Cons p ps e-B*)
let $?e-B' = (\text{fst } e-B, ps)$
let $?s = \text{last-state } ?e-B'$ **let** $?t = \text{snd } p$ **let** $?a = \text{fst } p$
have $1:\text{is-exec-of } B ?e-B'$ **and** $2:?s - ?a - B \longrightarrow ?t$
using *Cons.premis* **and** *Cons.hyps(2)*

by (*simp-all add:is-exec-of-def*,
 cases (B,fst e-B,ps#p) rule:is-exec-frag-of.cases, auto,
 cases (B,fst e-B,ps#p) rule:is-exec-frag-of.cases, auto)
 with *Cons.hyps(1)* obtain $e-A'$ where $ih1:is-exec-of A e-A'$
 and $ih2:trace (ioa.asig A) e-A' = trace (ioa.asig A) ?e-B'$
 and $ih3:last-state e-A' = f ?s$ by *fastforce*
 from 1 have $?3:reachable B ?s$ using *last-state-reachable* by *fast*
 obtain e where $?4:fst e = f ?s$ and $?5:last-state e = f ?t$
 and $?6:is-exec-frag-of A e$
 and $?7:let tr = trace (ioa.asig A) e$ in if $?a \in ext A$
 then $tr = [?a]$ else $tr = []$
 using 2 and 3 and *assms(1)*
 by (*force simp add:is-ref-map-def refines-def*)
 let $?e-A = append-exec e-A' e$
 have $is-exec-of A ?e-A$
 using $ih1 ih3 ?4 ?6$ *append-exec-frags-is-exec-frag[of A e e-A']*
 by (*metis append-exec-def append-exec-frags-is-exec-frag*
 fst-conv is-exec-of-def)
 moreover
 have $trace (ioa.asig A) ?e-A = trace (ioa.asig A) e-B$
 using $ih2$ *Cons.hyps(2)* $?7$ *step-eq-traces[of A e-A' ?e-B' ?a e]*
 by (*auto simp add:cons-exec-def*) (*metis prod.collapse*)
 moreover have $last-state ?e-A = f ?t$ using $ih3 ?4 ?5$ *last-state-of-append*
 by *metis*
 ultimately show $?case$ using *Cons.hyps(2)*
 by (*metis last-state.simps(2) surjective-pairing*)
 qed
 thus $?thesis$ by *blast*
 qed

theorem *ref-map-soundness*:

fixes $A::('sA,'a)ioa$ and $B::('sB,'a)ioa$ and $f::'sB \Rightarrow 'sA$
 assumes *is-ref-map f B A* and $ext A = ext B$
 shows $traces B \subseteq traces A$
 using *assms ref-map-execs exec-inc-imp-trace-inc* by *metis*

6.3 Soundness of Forward Simulations

lemma *forward-sim-execs*:

fixes $A::('sA,'a)ioa$ and $B::('sB,'a)ioa$ and $f::'sB \Rightarrow 'sA$ *set* and $e-B$
 assumes *is-forward-sim f B A* and *is-exec-of B e-B*
 shows $\exists e-A . is-exec-of A e-A$
 $\wedge trace (ioa.asig A) e-A = trace (ioa.asig A) e-B$

proof –

note *assms(2)*

hence $\exists e-A . is-exec-of A e-A$

$\wedge trace (ioa.asig A) e-A = trace (ioa.asig A) e-B$

$\wedge last-state e-A \in f (last-state e-B)$

proof (*induction snd e-B arbitrary:e-B*)

```

case Nil
have  $\bigwedge s . s \in \text{start } B \implies f s \cap \text{start } A \neq \{\}$ 
  using assms(1) by (simp add:is-forward-sim-def)
with this obtain s' where 1:s' ∈ f (fst e-B) and 2:s' ∈ start A
  by (metis Int-iff Nil.premis all-not-in-conv is-exec-of-def)
let ?e-A = (s', [])
have is-exec-of A ?e-A using 2 by (simp add:is-exec-of-def)
moreover
have trace (ioa.asig A) ?e-A = trace (ioa.asig A) e-B using Nil.hyps
  by (simp add:trace-def schedule-def filter-act-def)
moreover
have last-state ?e-A ∈ f (last-state e-B)
  using Nil.hyps 1 by (metis last-state.simps(1) surjective-pairing)
ultimately show ?case by fast
next
case (Cons p ps e-B)
let ?e-B' = (fst e-B, ps)
let ?s = last-state ?e-B' let ?t = snd p let ?a = fst p
have 1:is-exec-of B ?e-B' and 2: ?s - ?a - B → ?t
  using Cons.premis and Cons.hyps(2)
  by (simp-all add:is-exec-of-def,
      cases (B, fst e-B, ps#p) rule:is-exec-frag-of.cases, auto,
      cases (B, fst e-B, ps#p) rule:is-exec-frag-of.cases, auto)
with Cons.hyps(1) obtain e-A' where ih1:is-exec-of A e-A'
  and ih2:trace (ioa.asig A) e-A' = trace (ioa.asig A) ?e-B'
  and ih3:last-state e-A' ∈ f ?s by fastforce
from 1 have 3:reachable B ?s using last-state-reachable by fast
obtain e where 4:fst e = last-state e-A' and 5:last-state e ∈ f ?t
and 6:is-exec-frag-of A e
and 7:let tr = trace (ioa.asig A) e in if ?a ∈ ext A then tr = [?a] else tr = []
  using 2 3 assms(1) ih3 by (simp add:is-forward-sim-def)
  (metis prod.collapse prod.inject)
let ?e-A = append-exec e-A' e
have is-exec-of A ?e-A
  using ih1 ih3 4 6 append-exec-frags-is-exec-frag[of A e e-A']
  by (metis append-exec-def append-exec-frags-is-exec-frag
      fst-conv is-exec-of-def)
moreover
have trace (ioa.asig A) ?e-A = trace (ioa.asig A) e-B
  using ih2 Cons.hyps(2) 7 step-eq-traces[of A e-A' ?e-B' ?a e]
  by (auto simp add:cons-exec-def Let-def) (metis prod.collapse)
moreover have last-state ?e-A ∈ f ?t using ih3 4 5 last-state-of-append
  by metis
ultimately show ?case using Cons.hyps(2)
  by (metis last-state.simps(2) surjective-pairing)
qed
thus ?thesis by blast
qed

```


theorem *forward-sim-soundness*:
fixes $A::('sA, 'a)ioa$ **and** $B::('sB, 'a)ioa$ **and** $f::'sB \Rightarrow 'sA$ *set*
assumes *is-forward-sim* f B A **and** $ext\ A = ext\ B$
shows $traces\ B \subseteq traces\ A$
using *assms forward-sim-execs exec-inc-imp-trace-inc* **by** *metis*

6.4 Soundness of Backward Simulations

lemma *backward-sim-execs*:
fixes $A::('sA, 'a)ioa$ **and** $B::('sB, 'a)ioa$ **and** $f::'sB \Rightarrow 'sA$ *set* **and** $e-B$
assumes *is-backward-sim* f B A **and** *is-exec-of* B $e-B$
shows $\exists e-A . is-exec-of\ A\ e-A$
 $\wedge trace\ (ioa.asig\ A)\ e-A = trace\ (ioa.asig\ A)\ e-B$
proof –
note *assms(2)*
hence $\forall s \in f\ (last-state\ e-B) . \exists e-A .$
 $is-exec-of\ A\ e-A$
 $\wedge trace\ (ioa.asig\ A)\ e-A = trace\ (ioa.asig\ A)\ e-B$
 $\wedge last-state\ e-A = s$
proof (*induction snd e-B arbitrary:e-B*)
case *Nil*
{ **fix** s' **assume** $1:s' \in f(last-state\ e-B)$
have $2:\wedge s . s \in start\ B \implies f\ s \subseteq start\ A$
using *assms(1)* **by** (*simp add:is-backward-sim-def*)
from *Nil 1 2* **have** $3:s' \in start\ A$
by (*metis (full-types) is-exec-of-def last-state.simps(1) subsetD surjective-pairing*)
let $?e-A = (s', [])$
have $4:is-exec-of\ A\ ?e-A$ **using** 3 **by** (*simp add:is-exec-of-def*)
have $5:trace\ (ioa.asig\ A)\ ?e-A = trace\ (ioa.asig\ A)\ e-B$ **using** *Nil.hyps*
by (*simp add:trace-def schedule-def filter-act-def*)
have $6:last-state\ ?e-A \in f\ (last-state\ e-B)$
using *Nil.hyps 1* **by** (*metis last-state.simps(1)*)
note $4\ 5\ 6$ **}**
thus $?e-A$ **by** *fastforce*
next
case (*Cons p ps e-B*)
{ **fix** t' **assume** $8:t' \in f\ (last-state\ e-B)$
let $?e-B' = (fst\ e-B, ps)$
let $?s = last-state\ ?e-B'$ **let** $?t = snd\ p$ **let** $?a = fst\ p$
have $5:?t = last-state\ e-B$ **using** *Cons.hyps(2)*
by (*metis last-state.simps(2) prod.collapse*)
have $1:is-exec-of\ B\ ?e-B'$ **and** $2:?s - ?a - B \longrightarrow ?t$
using *Cons.prem* **and** *Cons.hyps(2)*
by (*simp-all add:is-exec-of-def,*
 $cases\ (B, fst\ e-B, ps\ \#p)$ *rule:is-exec-frag-of.cases, auto,*
 $cases\ (B, fst\ e-B, ps\ \#p)$ *rule:is-exec-frag-of.cases, auto*)
from 1 **have** $3:reachable\ B\ ?s$ **using** *last-state-reachable* **by** *fast*
obtain e **where** $4:fst\ e \in f\ ?s$ **and** $5:last-state\ e = t'$

```

and 6:is-exec-frag-of A e
and 7:let tr = trace (ioa.asig A) e in
  if ?a ∈ ext A then tr = [?a] else tr = []
  using 2 assms(1) 8 5 3 by (auto simp add: is-backward-sim-def, metis)
obtain e-A' where ih1:is-exec-of A e-A'
  and ih2:trace (ioa.asig A) e-A' = trace (ioa.asig A) ?e-B'
  and ih3:last-state e-A' = fst e
    using 1 4 Cons.hyps(1) by (metis snd-conv)
let ?e-A = append-exec e-A' e
have is-exec-of A ?e-A
  using ih1 ih3 4 6 append-exec-frags-is-exec-frag[of A e e-A']
  by (metis append-exec-def append-exec-frags-is-exec-frag
    fst-conv is-exec-of-def)
moreover
have trace (ioa.asig A) ?e-A = trace (ioa.asig A) e-B
  using ih2 Cons.hyps(2) 7 step-eq-traces[of A e-A' ?e-B' ?a e]
  by (auto simp add: cons-exec-def Let-def) (metis prod.collapse)
moreover have last-state ?e-A = t' using ih3 5 last-state-of-append
  by metis
ultimately have ∃ e-A . is-exec-of A e-A
  ∧ trace (ioa.asig A) e-A = trace (ioa.asig A) e-B
  ∧ last-state e-A = t' by blast }
thus ?case by blast
qed
moreover
from assms(1) have total:∧ s . f s ≠ {} by (simp add:is-backward-sim-def)
ultimately show ?thesis by fast
qed

```

```

theorem backward-sim-soundness:
  fixes A::('sA,'a)ioa and B::('sB,'a)ioa and f::'sB ⇒ 'sA set
  assumes is-backward-sim f B A and ext A = ext B
  shows traces B ⊆ traces A
  using assms backward-sim-execs exec-inc-imp-trace-inc by metis

```

end

end

7 Idempotence of the SLin I/O automaton

```

theory Idempotence
imports SLin Simulations
begin

locale Idempotence = SLin +
  fixes id1 id2 :: nat
  assumes id1:0 < id1 and id2:id1 < id2
begin

```

lemmas $ids = id1\ id2$

definition *composition* **where**

composition \equiv
 $hide\ ((ioa\ 0\ id1)\ ||\ (ioa\ id1\ id2))$
 $\{act . \exists p\ c\ av . act = Switch\ id1\ p\ c\ av\ \}$

lemmas $comp-simps = hide-def\ composition-def\ ioa-def\ par2-def\ is-trans-def$
 $start-def\ actions-def\ asig-def\ trans-def$

lemmas $trans-defs = Inv-def\ Lin-def\ Resp-def\ Init-def$
 $Abort-def\ Reco-def$

declare *if-split-asm* [*split*]

7.1 A case rule for decomposing the transition relation of the composition of two SLins

declare *comp-simps* [*simp*]

lemma *trans-elim*:

fixes $s\ t\ a\ s'\ t'\ P$

assumes $(s,t) -a-composition \longrightarrow (s',t')$

obtains

$(Invoke1)\ i\ p\ c$

where $Inv\ p\ c\ s\ s' \wedge t = t'$

and $i < id1$ **and** $a = Invoke\ i\ p\ c$

| $(Invoke2)\ i\ p\ c$

where $Inv\ p\ c\ t\ t' \wedge s = s'$

and $id1 \leq i \wedge i < id2$ **and** $a = Invoke\ i\ p\ c$

| $(Switch1)\ p\ c\ av$

where $Abort\ p\ c\ av\ s\ s' \wedge Init\ p\ c\ av\ t\ t'$

and $a = Switch\ id1\ p\ c\ av$

| $(Switch2)\ p\ c\ av$

where $s = s' \wedge Abort\ p\ c\ av\ t\ t'$

and $a = Switch\ id2\ p\ c\ av$

| $(Response1)\ i\ p\ ou$

where $Resp\ p\ ou\ s\ s' \wedge t = t'$

and $i < id1$ **and** $a = Response\ i\ p\ ou$

| $(Response2)\ i\ p\ ou$

where $Resp\ p\ ou\ t\ t' \wedge s = s'$

and $id1 \leq i \wedge i < id2$ **and** $a = Response\ i\ p\ ou$

| $(Lin1)\ Lin\ s\ s' \wedge t = t'$ **and** $a = Linearize\ 0$

| $(Lin2)\ Lin\ t\ t' \wedge s = s'$ **and** $a = Linearize\ id1$

| $(Reco2)\ Reco\ t\ t' \wedge s = s'$ **and** $a = Recover\ id1$

declare *comp-simps* [*simp del*]

7.2 Definition of the Refinement Mapping

fun $f :: (('a,'b,'c)SLin\text{-state} * ('a,'b,'c)SLin\text{-state}) \Rightarrow ('a,'b,'c)SLin\text{-state}$
where
 $f (s1, s2) =$
 ($pending = \lambda p. (if\ status\ s1\ p \neq\ Aborted\ then\ pending\ s1\ p\ else\ pending\ s2\ p)$),
 $initVals = \{\}$,
 $abortVals = abortVals\ s2$,
 $status = \lambda p. (if\ status\ s1\ p \neq\ Aborted\ then\ status\ s1\ p\ else\ status\ s2\ p)$),
 $dstate = (if\ dstate\ s2 = \perp\ then\ dstate\ s1\ else\ dstate\ s2)$,
 $initialized = True$)

7.3 Invariants

declare

$trans\text{-}defs\ [simp]$

fun $P1$ **where**

$P1 (s1, s2) = (\forall p. status\ s1\ p \in \{Pending, Aborted\}$
 $\longrightarrow fst (pending\ s1\ p) = p)$

fun $P2$ **where**

$P2 (s1, s2) = (\forall p. status\ s2\ p \neq Sleep \longrightarrow fst (pending\ s2\ p) = p)$

fun $P3$ **where**

$P3 (s1, s2) = (\forall p. (status\ s2\ p = Ready \longrightarrow initialized\ s2))$

fun $P4$ **where**

$P4 (s1, s2) = ((\forall p. status\ s2\ p = Sleep) = (initVals\ s2 = \{\}))$

fun $P5$ **where**

$P5 (s1, s2) = (\forall p. status\ s1\ p \neq Sleep \wedge initialized\ s1 \wedge initVals\ s1 = \{\})$

fun $P6$ **where**

$P6 (s1, s2) = (\forall p. (status\ s1\ p \neq Aborted) = (status\ s2\ p = Sleep))$

fun $P7$ **where**

$P7 (s1, s2) = (\forall c. status\ s1\ c = Aborted \wedge \neg initialized\ s2$
 $\longrightarrow (pending\ s2\ c = pending\ s1\ c \wedge status\ s2\ c \in \{Pending, Aborted\}))$

fun $P8$ **where**

$P8 (s1, s2) = (\forall iv \in initVals\ s2. \exists rs \in pendingSeqs\ s1.$
 $iv = dstate\ s1 \star rs)$

fun $P8a$ **where**

$P8a (s1, s2) = (\forall ivs \in initSets\ s2. \exists rs \in pendingSeqs\ s1.$
 $\sqcap ivs = dstate\ s1 \star rs)$

fun P9 where

$$P9 (s1,s2) = (\text{initialized } s2 \longrightarrow \text{dstate } s1 \preceq \text{dstate } s2)$$

fun P10 where

$$P10 (s1,s2) = ((\neg \text{initialized } s2) \longrightarrow (\text{dstate } s2 = \perp))$$

fun P11 where

$$P11 (s1,s2) = (\text{initVals } s2 = \text{abortVals } s1)$$

fun P12 where

$$P12 (s1,s2) = (\text{initialized } s2 \longrightarrow \sqcap (\text{initVals } s2) \preceq \text{dstate } s2)$$

fun P13 where

$$P13 (s1,s2) = (\text{finite } (\text{initVals } s2) \\ \wedge \text{finite } (\text{abortVals } s1) \wedge \text{finite } (\text{abortVals } s2))$$

fun P14 where

$$P14 (s1,s2) = (\text{initialized } s2 \longrightarrow \text{initVals } s2 \neq \{\})$$

fun P15 where

$$P15 (s1,s2) = (\forall av \in \text{abortVals } s1 . \text{dstate } s1 \preceq av)$$

fun P16 where

$$P16 (s1,s2) = (\text{dstate } s2 \neq \perp \longrightarrow \text{initialized } s2)$$

fun P17 where

— For the Response1 case of the refinement proof, in case a response is produced in the first instance and the second instance is already initialized

$$P17 (s1,s2) = (\text{initialized } s2 \\ \longrightarrow (\forall p . \\ ((\text{status } s1 p = \text{Ready} \\ \vee (\text{status } s1 p = \text{Pending} \wedge \text{contains } (\text{dstate } s1) (\text{pending } s1 p))) \\ \longrightarrow (\exists rs . \text{dstate } s2 = \text{dstate } s1 \star rs \wedge (\forall r \in \text{set } rs . \text{fst } r \neq p))) \\ \wedge ((\text{status } s1 p = \text{Pending} \wedge \neg \text{contains } (\text{dstate } s1) (\text{pending } s1 p)) \\ \longrightarrow (\exists rs . \text{dstate } s2 = \text{dstate } s1 \star rs \wedge (\forall r \in \text{set } rs . \\ \text{fst } r = p \longrightarrow r = \text{pending } s1 p))))))$$

fun P18 where

$$P18 (s1,s2) = (\text{abortVals } s2 \neq \{\} \longrightarrow (\exists p . \text{status } s2 p \neq \text{Sleep}))$$

fun P19 where

$$P19 (s1,s2) = (\text{abortVals } s2 \neq \{\} \longrightarrow \text{abortVals } s1 \neq \{\})$$

fun P20 where

$$P20 (s1,s2) = (\forall av \in \text{abortVals } s2 . \text{dstate } s2 \preceq av)$$

fun P21 where

$P21 (s1,s2) = (\forall av \in abortVals s2 . \prod (abortVals s1) \preceq av)$

fun P22 where

$P22 (s1,s2) = (initialized s2 \longrightarrow dstate (f (s1,s2)) = dstate s2)$

fun P23 where

$P23 (s1,s2) = ((\neg initialized s2) \longrightarrow$
 $pendingSeqs s1 \subseteq pendingSeqs (f (s1,s2)))$

fun P25 where

$P25 (s1,s2) = (\forall ivs . (ivs \in initSets s2 \wedge initialized s2$
 $\wedge dstate s2 \preceq \prod ivs)$
 $\longrightarrow (\exists rs' \in pendingSeqs (f (s1,s2)) . \prod ivs = dstate s2 \star rs'))$

fun P26 where

$P26 (s1,s2) = (\forall p . (status s1 p = Aborted$
 $\wedge \neg contains (dstate s2) (pending s1 p))$
 $\longrightarrow (status s2 p \in \{Pending, Aborted\}$
 $\wedge pending s1 p = pending s2 p))$

lemma P1-invariant:

shows invariant (composition) P1

proof (rule invariantI, simp-all only:split-paired-all)

fix s1 s2

assume (s1,s2) ∈ ioa.start (composition)

thus P1 (s1,s2) using ids by (auto simp add:comp-simps)

next

fix s1 s2 t1 t2 a

assume hyp: P1 (s1,s2) and trans:(s1,s2) -a-composition→ (t1,t2)

show P1 (t1,t2) using trans and hyp

by (cases rule:trans-elim) auto

qed

lemma P2-invariant:

shows invariant (composition) P2

proof (rule invariantI, simp-all only:split-paired-all)

fix s1 s2

assume (s1,s2) ∈ ioa.start (composition)

thus P2 (s1,s2) using ids by (auto simp add:comp-simps)

next

fix s1 s2 t1 t2 a

assume hyp: P2 (s1,s2) and trans:(s1,s2) -a-composition→ (t1,t2)

show P2 (t1,t2) using trans and hyp

by (cases rule:trans-elim) auto

qed

lemma P16-invariant:

shows invariant (composition) P16

proof (rule invariantI, simp-all only:split-paired-all)

```

fix  $s1\ s2$ 
assume  $(s1,s2) \in ioa.start\ (composition)$ 
thus  $P16\ (s1,s2)$  using  $ids$  by  $(auto\ simp\ add:comp-simps)$ 
next
fix  $s1\ s2\ t1\ t2\ a$ 
assume  $hyp: P16\ (s1,s2)$  and  $trans:(s1,s2) -a-composition \longrightarrow (t1,t2)$ 
show  $P16\ (t1,t2)$  using  $trans$  and  $hyp$ 
by  $(cases\ rule:trans-elim)$   $auto$ 
qed

```

```

lemma  $P3$ -invariant:
shows  $invariant\ (composition)\ P3$ 
proof  $(rule\ invariantI,\ simp-all\ only:split-paired-all)$ 
fix  $s1\ s2$ 
assume  $(s1,s2) \in ioa.start\ (composition)$ 
thus  $P3\ (s1,s2)$  using  $ids$  by  $(auto\ simp\ add:comp-simps)$ 
next
fix  $s1\ s2\ t1\ t2\ a$ 
assume  $hyp: P3\ (s1,s2)$  and  $trans:(s1,s2) -a-composition \longrightarrow (t1,t2)$ 
show  $P3\ (t1,t2)$  using  $trans$  and  $hyp$ 
by  $(cases\ rule:trans-elim)$   $auto$ 
qed

```

```

lemma  $P4$ -invariant:
shows  $invariant\ (composition)\ P4$ 
proof  $(rule\ invariantI,\ simp-all\ only:split-paired-all)$ 
fix  $s1\ s2$ 
assume  $(s1,s2) \in ioa.start\ (composition)$ 
thus  $P4\ (s1,s2)$  using  $ids$  by  $(auto\ simp\ add:comp-simps)$ 
next
fix  $s1\ s2\ t1\ t2\ a$ 
assume  $hyp: P4\ (s1,s2)$  and  $trans:(s1,s2) -a-composition \longrightarrow (t1,t2)$ 
show  $P4\ (t1,t2)$  using  $trans$  and  $hyp$ 
by  $(cases\ rule:trans-elim)$   $auto$ 
qed

```

```

lemma  $P5$ -invariant:
shows  $invariant\ (composition)\ P5$ 
proof  $(rule\ invariantI,\ simp-all\ only:split-paired-all)$ 
fix  $s1\ s2$ 
assume  $(s1,s2) \in ioa.start\ (composition)$ 
thus  $P5\ (s1,s2)$  using  $ids$  by  $(auto\ simp\ add:comp-simps)$ 
next
fix  $s1\ s2\ t1\ t2\ a$ 
assume  $hyp: P5\ (s1,s2)$  and  $trans:(s1,s2) -a-composition \longrightarrow (t1,t2)$ 
show  $P5\ (t1,t2)$  using  $trans$  and  $hyp$ 
by  $(cases\ rule:trans-elim)$   $auto$ 
qed

```

```

lemma P13-invariant:
shows invariant (composition) P13
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume  $(s1,s2) \in \text{ioa.start (composition)}$ 
  thus P13 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P13 (s1,s2) and trans:(s1,s2) -a-composition $\longrightarrow$  (t1,t2)
  show P13 (t1,t2) using trans and hyp
  by (cases rule:trans-elim, auto)
qed

```

```

lemma P20-invariant:
shows invariant (composition) P20
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume  $(s1,s2) \in \text{ioa.start (composition)}$ 
  thus P20 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P20 (s1,s2) and trans:(s1,s2) -a-composition $\longrightarrow$  (t1,t2)
  and reach: reachable (composition) (s1,s2)
  from reach have P16:P16 (s1,s2) using P16-invariant and ids
  by (metis IOA.invariant-def)
  show P20 (t1,t2) using trans and hyp and P16
  by (cases rule:trans-elim, auto simp add:safeInits-def safeAborts-def
    initAborts-def uninitAborts-def bot)
qed

```

```

lemma P18-invariant:
shows invariant (composition) P18
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume  $(s1,s2) \in \text{ioa.start (composition)}$ 
  thus P18 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P18 (s1,s2) and trans:(s1,s2) -a-composition $\longrightarrow$  (t1,t2)
  show P18 (t1,t2) using trans and hyp
  by (cases rule:trans-elim, auto)
qed

```

```

lemma P14-invariant:
shows invariant (composition) P14
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume  $(s1,s2) \in \text{ioa.start (composition)}$ 

```



```

  thus P14 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P14 (s1,s2) and trans:(s1,s2) -a-composition⟶ (t1,t2)
  show P14 (t1,t2) using trans and hyp
  by (cases rule:trans-elim, auto simp add:safeInits-def)
qed

```

```

lemma P15-invariant:
shows invariant (composition) P15
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume (s1,s2) ∈ ioa.start (composition)
  thus P15 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P15 (s1,s2) and trans:(s1,s2) -a-composition⟶ (t1,t2)
  and reach: reachable (composition) (s1,s2)
  from reach have P5:P5 (s1,s2) using P5-invariant and ids
  by (metis IOA.invariant-def)
  show P15 (t1,t2) using trans and hyp and P5
  by (cases rule:trans-elim,
      auto simp add:less-eq-def safeAborts-def initAborts-def)
qed

```

```

lemma P6-invariant:
shows invariant (composition) P6
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume (s1,s2) ∈ ioa.start (composition)
  thus P6 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P6 (s1,s2) and trans:(s1,s2) -a-composition⟶ (t1,t2)
  show P6 (t1,t2) using trans and hyp
  by (cases rule:trans-elim, force+)
qed

```

```

lemma P7-invariant:
shows invariant (composition) P7
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume (s1,s2) ∈ ioa.start (composition)
  thus P7 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P7 (s1,s2) and trans:(s1,s2) -a-composition⟶ (t1,t2)
  show P7 (t1,t2) using trans and hyp
  by (cases rule:trans-elim) auto

```

qed

lemma *P10-invariant*:

shows *invariant (composition) P10*

proof (rule *invariantI*, simp-all only:*split-paired-all*)

fix *s1 s2*

assume $(s1,s2) \in \text{ioa.start (composition)}$

thus *P10 (s1,s2) using ids by (auto simp add:comp-simps)*

next

fix *s1 s2 t1 t2 a*

assume *hyp: P10 (s1,s2) and trans:(s1,s2) -a-composition → (t1,t2)*

show *P10 (t1,t2) using trans and hyp*

by (cases rule:*trans-elim*) auto

qed

lemma *P11-invariant*:

shows *invariant (composition) P11*

proof (rule *invariantI*, simp-all only:*split-paired-all*)

fix *s1 s2*

assume $(s1,s2) \in \text{ioa.start (composition)}$

thus *P11 (s1,s2) using ids by (auto simp add:comp-simps)*

next

fix *s1 s2 t1 t2 a*

assume *hyp: P11 (s1,s2) and trans:(s1,s2) -a-composition → (t1,t2)*

show *P11 (t1,t2) using trans and hyp*

by (cases rule:*trans-elim, force+*)

qed

lemma *P8-invariant*:

shows *invariant (composition) P8*

proof (rule *invariantI*, simp-all only:*split-paired-all*)

fix *s1 s2*

assume $(s1,s2) \in \text{ioa.start (composition)}$

thus *P8 (s1,s2) using ids by (auto simp add:comp-simps)*

next

fix *s1 s2 t1 t2 a*

assume *hyp: P8 (s1,s2) and trans:(s1,s2) -a-composition → (t1,t2)*

and *reach: reachable (composition) (s1,s2)*

from *reach have P5:P5 (s1,s2) using P5-invariant and ids*

by (metis *IOA.invariant-def*)

from *reach have P1:P1 (s1,s2) using P1-invariant and ids*

by (metis *IOA.invariant-def*)

from *reach have P11:P11 (s1,s2) using P11-invariant and ids*

by (metis *IOA.invariant-def*)

show *P8 (t1,t2) using trans and hyp*

proof (cases rule:*trans-elim*)

case (*Invoke1 i p c*)

assume *P8 (s1,s2)*

have *pendingSeqs s1 ⊆ pendingSeqs t1*

```

proof –
  have pending t1 = (pending s1)(p := (p,c))
  and status t1 = (status s1)(p := Pending)
  and status s1 p = Ready
  using Invoke1(1) by auto
  hence pendingReqs s1  $\subseteq$  pendingReqs t1 by (simp add:pendingReqs-def) force
  thus ?thesis by (auto simp add:pendingSeqs-def)
qed
moreover have initVals t2 = initVals s2 and dstate t1 = dstate s1
  using Invoke1(1) by auto
ultimately show P8 (t1,t2) using  $\langle P8 (s1,s2) \rangle$  by fastforce
next
case Lin1
assume P8 (s1,s2)
show P8 (t1,t2)
proof (simp, rule ballI)
  fix iv
  assume  $0:iv \in \text{initVals } t2$ 
  have  $1:iv \in \text{initVals } s2$  using Lin1(1) 0 by simp
  have  $4:iv \in \text{abortVals } s1$  using 1 P11 by simp
  obtain rs where  $2:rs \in \text{pendingSeqs } s1$  and  $3:iv = \text{dstate } s1 \star rs$ 
    using  $\langle P8 (s1,s2) \rangle 1$  by auto
  obtain rs' where  $6:\text{dstate } t1 = \text{dstate } s1 \star rs'$  and  $5:\text{dstate } s1 \star rs' \preceq iv$ 
    using Lin1(1) 1 4 by auto
  obtain rs'' where  $7:iv = (\text{dstate } s1 \star rs') \star rs''$  and  $8:\text{set } rs'' \subseteq \text{set } rs$ 
    using consistency 3 5 6 by simp (metis less-eq-def)
  have  $10:rs'' \in \text{pendingSeqs } t1$ 
  proof –
    have  $9:\text{pendingSeqs } t1 = \text{pendingSeqs } s1$ 
      using Lin1(1) by (auto simp add:pendingSeqs-def pendingReqs-def)
    thus ?thesis using 8 2 by (auto simp add:pendingSeqs-def)
  qed
  show  $\exists rs \in \text{pendingSeqs } t1 . iv = \text{dstate } t1 \star rs$ 
    using 7 10 6 by auto
qed
next
case (Response1 i p ou)
assume ih:P8 (s1,s2)
show P8 (t1,t2)
proof auto
  fix iv
  assume  $1:iv \in \text{initVals } t2$ 
  obtain rs where  $2:iv = \text{dstate } t1 \star rs$  and  $3:rs \in \text{pendingSeqs } s1$ 
    using 1 Response1(1) ih by auto
  have  $4:\text{pendingReqs } t1 = ((\text{pendingReqs } s1) - \{\text{pending } s1 p\})$ 
  proof –
    have pending t1 = pending s1 and status t1 = (status s1)(p := Ready)
      and  $5:\text{status } s1 p = \text{Pending}$ 
    using Response1(1) by auto
  qed

```

```

moreover have  $\bigwedge q . q \neq p \implies \text{status } s1 \ q \in \{\text{Pending, Aborted}\}$ 
   $\implies \text{pending } s1 \ q \neq \text{pending } s1 \ p$ 
  using P1 5 by (metis P1.simps insertI1)
ultimately show ?thesis by (simp add:pendingReqs-def) fastforce
qed
have 8:contains (dstate t1) (pending s1 p) using Response1(1) by simp
define rs' where rs' = filter (λ x . x ≠ (pending s1 p)) rs
have 9:rs' ∈ pendingSeqs t1
proof –
  have 9:pending s1 p ∉ set rs' by (auto simp add:rs'-def)
  have 10:rs' ∈ pendingSeqs s1
    using 3 by (auto simp add:rs'-def)
    (metis filter-is-subset mem-Collect-eq pendingSeqs-def subset-trans)
  show ?thesis using 10 9 4 by (auto simp add:pendingSeqs-def)
qed
have 10:iv = dstate t1 ★ rs' using 8 2 idem-star rs'-def by fast
show  $\exists rs \in \text{pendingSeqs } t1 . iv = \text{dstate } t1 \star rs$  using 10 9 by auto
qed
next
case (Switch1 p c av)
assume P8 (s1,s2)
have 1:initialized s1 ∧ initVals s1 = {} using P5 by auto
obtain av where 2:initVals t2 = initVals s2 ∪ {av} and 3:av ∈ safeAborts s1
  using Switch1(1) by auto
obtain rs where 4:rs ∈ pendingSeqs s1 and 5:av = dstate s1 ★ rs
  using 1 3 by (auto simp add:safeAborts-def initAborts-def initSets-def)
have 6:dstate s1 = dstate t1 using Switch1(1) by simp
have 7:pendingSeqs t1 = pendingSeqs s1
proof –
  have pendingReqs t1 = pendingReqs s1
    using Switch1(1) by (simp add:pendingReqs-def) fastforce
  thus ?thesis by (auto simp add:pendingSeqs-def)
qed
show P8 (t1,t2) using  $\langle P8 (s1,s2) \rangle$  2 4 5 6 7 by auto
next
case (Invoke2 i p c)
assume P8 (s1,s2)
thus P8 (t1,t2) using Invoke2(1) by force
next
case Lin2
assume P8 (s1,s2)
thus P8 (t1,t2) using Lin2(1) by force
next
case (Response2 i p ou)
assume P8 (s1,s2)
thus P8 (t1,t2) using Response2(1) by force
next
case (Switch2 p c av)
assume P8 (s1,s2)

```

```

    thus P8 (t1,t2) using Switch2(1) by force
next
  case Reco2
  assume P8 (s1,s2)
  thus P8 (t1,t2) using Reco2(1) by force
qed
qed

lemma P8a-invariant:
shows invariant (composition) P8a
proof (auto simp:invariant-def)
  fix s1 s2 ivs
  assume 1:reachable (composition) (s1,s2)
  and 2:ivs ∈ initSets s2
  have 3:finite ivs ∧ ivs ≠ {}
  proof -
    have P13 (s1,s2) using P13-invariant 1
      by (metis IOA.invariant-def)
    thus ?thesis using 2 finite-subset by (auto simp add:initSets-def)
  qed
  have 4:∀ av ∈ ivs . ∃ rs ∈ pendingSeqs s1 . av = dstate s1 ★ rs
  proof -
    have P8:P8 (s1,s2) using P8-invariant 1
      by (metis IOA.invariant-def)
    thus ?thesis using 2 by (auto simp add:initSets-def)
  qed
  show ∃ rs ∈ pendingSeqs s1 . ⋂ ivs = dstate s1 ★ rs
    using 3 4 glb-common-set by (simp add:pendingSeqs-def, metis)
qed

```

```

lemma P12-invariant:
shows invariant (composition) P12
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume (s1,s2) ∈ ioa.start (composition)
  thus P12 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P12 (s1,s2) and trans:(s1,s2) -a-composition→ (t1,t2)
  and reach: reachable (composition) (s1,s2)
  from reach have P13:P13 (s1,s2) using P13-invariant
    by (metis IOA.invariant-def)
  from reach have P14:P14 (s1,s2) using P14-invariant
    by (metis IOA.invariant-def)
  show P12 (t1,t2) using trans and hyp
  proof (cases rule:trans-elim)
    case (Invoke1 i p c)
    assume P12 (s1,s2)
    thus P12 (t1,t2) using Invoke1(1) by auto
  qed

```

```

next
  case Lin1
  assume P12 (s1,s2)
  thus P12 (t1,t2) using Lin1(1) by auto
next
  case (Response1 i p ou)
  assume P12 (s1,s2)
  thus P12 (t1,t2) using Response1(1) by auto
next
  case (Switch1 p c av)
  assume ih:P12 (s1,s2)
  have initialized s2  $\implies \sqcap$  (initVals t2)  $\preceq \sqcap$  (initVals s2)
  proof -
    assume 1:initialized s2
    have initVals t2 = initVals s2  $\cup$  {av} using Switch1(1) by simp
    hence  $\sqcap$  (initVals t2) =  $\sqcap$  (initVals s2)  $\sqcap$  av
      using insert-not-elem P13 P14 1
      by (metis P13.simps P14.simps Un-empty-right Un-insert-right commute
insert)
    thus ?thesis by (metis cobounded1)
  qed
  moreover have dstate t2 = dstate s2 and initialized s2 = initialized t2
    using Switch1(1) by auto
  ultimately show P12 (t1,t2) using ih by auto (metis absorb2 coboundedI1)
next
  case (Invoke2 i p c)
  assume P12 (s1,s2)
  thus P12 (t1,t2) using Invoke2(1) by force
next
  case Lin2
  assume P12 (s1,s2)
  moreover
  have initVals t2 = initVals s2 and initialized s2
  and initialized t2 using Lin2(1) by auto
  moreover
  have dstate s2  $\preceq$  dstate t2 using Lin2(1) by auto (metis less-eq-def)
  ultimately show P12 (t1,t2) by auto (metis strict-iff-order strict-trans1)
next
  case (Response2 i p ou)
  assume P12 (s1,s2)
  thus P12 (t1,t2) using Response2(1) by force
next
  case (Switch2 p c av)
  assume P12 (s1,s2)
  thus P12 (t1,t2) using Switch2(1) by force
next
  case Reco2
  obtain d where 1:d  $\in$  safeInits s2 and 2:dstate t2 = d
    using Reco2(1) by force

```

```

obtain ivs where  $3:ivs \subseteq \text{initVals } s2$  and  $4:ivs \neq \{\}$ 
and  $5:\sqcap ivs \preceq d$ 
using 1 by (auto simp add:safeInits-def initSets-def)
  (metis equals0D less-eq-def)
have  $6:\sqcap (\text{initVals } s2) \preceq \sqcap ivs$  using 3 P13 4
by (simp add: subset-imp)
have  $7:\text{initVals } s2 = \text{initVals } t2$  using Reco2(1) by auto
show P12 (t1,t2) using 2 5 6 7
by (metis P12.simps absorb2 coboundedI1)
qed
qed

```

```

lemma P19-invariant:
shows invariant (composition) P19
proof (auto simp only:invariant-def)
  fix s1 s2
  assume  $1:\text{reachable (composition) (s1,s2)}$ 
  have  $P4:P4 (s1,s2)$  using P4-invariant 1
  by (simp add:invariant-def)
  moreover
  have  $P18:P18 (s1,s2)$  using P18-invariant 1
  by (metis IOA.invariant-def)
  moreover
  have  $P11:P11 (s1,s2)$  using P11-invariant 1
  by (metis IOA.invariant-def)
  moreover
  ultimately show P19 (s1,s2) by auto
qed

```

```

lemma P9-invariant:
shows invariant (composition) P9
proof (auto simp only:invariant-def)
  fix s1 s2
  assume  $1:\text{reachable (composition) (s1,s2)}$ 
  have  $P12:P12 (s1,s2)$  using P12-invariant 1
  by (simp add:invariant-def)
  have  $P15:P15 (s1,s2)$  using P15-invariant 1
  by (metis IOA.invariant-def)
  have  $P13:P13 (s1,s2)$  using P13-invariant 1
  by (metis IOA.invariant-def)
  have  $P14:P14 (s1,s2)$  using P14-invariant 1
  by (metis IOA.invariant-def)
  have  $P11:P11 (s1,s2)$  using P11-invariant 1
  by (metis IOA.invariant-def)
  have  $\text{initialized } s2 \implies \text{dstate } s1 \preceq \sqcap (\text{abortVals } s1)$ 
  using P13 P15 P14 P11 boundedI by simp
  thus P9 (s1,s2) using P12 P11 by simp (metis trans)
qed

```

```

lemma P17-invariant:
shows invariant (composition) P17
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume (s1,s2) ∈ ioa.start (composition)
  thus P17 (s1,s2) using ids by (auto simp add:comp-simps)
next
  fix s1 s2 t1 t2 a
  assume hyp: P17 (s1,s2) and trans:(s1,s2) -a-composition→ (t1,t2)
  and reach:reachable (composition) (s1,s2)
  show P17 (t1,t2) using trans and hyp
  proof (cases rule:trans-elim)
    case (Invoke1 i p c)
    assume P17 (s1,s2)
    thus P17 (t1,t2) using Invoke1(1) by fastforce
  next
    case (Response1 i p ou)
    assume P17 (s1,s2)
    thus P17 (t1,t2) using Response1(1) by auto
  next
    case (Switch1 p c av)
    assume P17 (s1,s2)
    thus P17 (t1,t2) using Switch1(1) by auto
  next
    case (Invoke2 i p c)
    assume P17 (s1,s2)
    thus P17 (t1,t2) using Invoke2(1) by force
  next
    case (Response2 i p ou)
    assume P17 (s1,s2)
    thus P17 (t1,t2) using Response2(1) by force
  next
    case (Switch2 p c av)
    assume P17 (s1,s2)
    thus P17 (t1,t2) using Switch2(1) by force
  next
    case Lin1
    assume 1:P17 (s1,s2)
    obtain rs' where 2:dstate t1 = dstate s1 ★ rs'
      using Lin1(1) 1 by auto
    have 3:dstate s2 = dstate t2 using Lin1(1) by auto
    have 4:initialized t2 ⇒ dstate t1 ≼ dstate t2
    proof -
      assume initialized t2
      moreover
      have P9 (t1,t2) using reach trans P9-invariant
        by (metis IOA.invariant-def reachable-n)
      ultimately show ?thesis by auto
    qed
  qed

```



```

show P17 (t1,t2)
proof(simp, auto)
  fix p
  assume 5:initialized t2 and 6:status t1 p = Ready
  obtain rs where 7: $\forall r \in \text{set } rs . \text{fst } r \neq p$ 
    and 8:dstate t2 = dstate s1  $\star$  rs
  proof -
    obtain rs where dstate s2 = dstate s1  $\star$  rs
       $\wedge (\forall r \in \text{set } rs . \text{fst } r \neq p)$  using 1 5 6 Lin1(1) by force
    hence  $\forall r \in \text{set } rs . \text{fst } r \neq p$  and dstate t2 = dstate s1  $\star$  rs
      using Lin1(1) by auto
    thus ?thesis using that by blast
  qed
  have 9:dstate t1  $\preceq$  dstate t2 using 4 5 by auto
  obtain rs'' where 10:dstate t2 = dstate t1  $\star$  rs''
    and 11:set rs''  $\subseteq$  set rs
    using consistency 2 8 9 by simp (metis less-eq-def)
  have 12: $\forall r \in \text{set } rs'' . \text{fst } r \neq p$  using 7 11 by blast
  thus  $\exists rs . \text{dstate } t2 = \text{dstate } t1 \star rs \wedge (\forall r \in \text{set } rs . \text{fst } r \neq p)$ 
    using 10 12 by auto
next
fix p
assume 5:initialized t2 and 6:status t1 p = Pending
  and 7: $\neg \text{contains } (\text{dstate } t1) (\text{pending } t1 p)$ 
  obtain rs where 8: $\forall r \in \text{set } rs . \text{fst } r = p \longrightarrow r = \text{pending } s1 p$ 
    and 9:dstate t2 = dstate s1  $\star$  rs
  proof -
    have 9: $\neg \text{contains } (\text{dstate } s1) (\text{pending } s1 p)$ 
      using 7 Lin1(1) contains-star by fastforce
    obtain rs where dstate s2 = dstate s1  $\star$  rs
       $\wedge (\forall r \in \text{set } rs . \text{fst } r = p \longrightarrow r = \text{pending } s1 p)$ 
      using 1 5 6 9 Lin1(1) by force
    hence  $\forall r \in \text{set } rs . \text{fst } r = p \longrightarrow r = \text{pending } s1 p$ 
      and dstate t2 = dstate s1  $\star$  rs
      using Lin1(1) by auto
    thus ?thesis using that by blast
  qed
  have 10:dstate t1  $\preceq$  dstate t2 using 4 5 by auto
  obtain rs'' where 11:dstate t2 = dstate t1  $\star$  rs''
    and 12:set rs''  $\subseteq$  set rs
    using consistency 2 9 10 by simp (metis less-eq-def)
  have 13: $\forall r \in \text{set } rs'' . \text{fst } r = p \longrightarrow r = \text{pending } s1 p$ 
    using 8 12 by blast
  show  $\exists rs . \text{dstate } t2 = \text{dstate } t1 \star rs$ 
     $\wedge (\forall r \in \text{set } rs . \text{fst } r = p \longrightarrow r = \text{pending } t1 p)$ 
    using 11 13 Lin1(1) by auto
next
fix p
assume 5:initialized t2 and 6:status t1 p = Pending

```

```

    and 7:contains (dstate t1) (pending t1 p)
show  $\exists rs . dstate t2 = dstate t1 \star rs$ 
 $\wedge (\forall r \in set rs . fst r \neq p)$ 
proof (cases contains (dstate s1) (pending s1 p))
case True
obtain rs where 8: $\forall r \in set rs . fst r \neq p$ 
and 9:dstate t2 = dstate s1  $\star$  rs
proof -
obtain rs where dstate s2 = dstate s1  $\star$  rs
 $\wedge (\forall r \in set rs . fst r \neq p)$  using 1 5 6 True Lin1(1) by force
hence  $\forall r \in set rs . fst r \neq p$  and dstate t2 = dstate s1  $\star$  rs
using Lin1(1) by auto
thus ?thesis using that by blast
qed
have 10:dstate t1  $\preceq$  dstate t2 using 4 5 by auto
obtain rs'' where 11:dstate t2 = dstate t1  $\star$  rs''
and 12:set rs''  $\subseteq$  set rs
using consistency 2 9 10 by simp (metis less-eq-def)
have 13: $\forall r \in set rs'' . fst r \neq p$  using 8 12 by blast
thus  $\exists rs . dstate t2 = dstate t1 \star rs \wedge (\forall r \in set rs . fst r \neq p)$ 
using 11 13 by auto
next
case False
obtain rs'' where 8:dstate t2 = dstate t1  $\star$  rs''
and 9: $\forall r \in set rs'' . fst r = p \longrightarrow r = pending t1 p$ 
proof -
obtain rs where 8: $\forall r \in set rs . fst r = p \longrightarrow r = pending s1 p$ 
and 9:dstate t2 = dstate s1  $\star$  rs
proof -
obtain rs where dstate s2 = dstate s1  $\star$  rs
 $\wedge (\forall r \in set rs . fst r = p \longrightarrow r = pending s1 p)$ 
using 1 5 6 False Lin1(1) by force
hence  $\forall r \in set rs . fst r = p \longrightarrow r = pending s1 p$ 
and dstate t2 = dstate s1  $\star$  rs
using Lin1(1) by auto
thus ?thesis using that by blast
qed
have 10:dstate t1  $\preceq$  dstate t2 using 4 5 by auto
obtain rs'' where 11:dstate t2 = dstate t1  $\star$  rs''
and 12:set rs''  $\subseteq$  set rs
using consistency 2 9 10 by simp (metis less-eq-def)
have 13: $\forall r \in set rs'' . fst r = p \longrightarrow r = pending s1 p$ 
using 8 12 by blast
have dstate t2 = dstate t1  $\star$  rs''
 $\wedge (\forall r \in set rs'' . fst r = p \longrightarrow r = pending t1 p)$ 
using 11 13 Lin1(1) by auto
thus ?thesis using that by blast
qed
have 10:dstate t1  $\star$  rs''

```

```

    = dstate t1 * (filter (λ r . r ≠ pending t1 p) rs'')
    using 7 idem-star by blast
  have 11:∀ r ∈ set (filter (λ r . r ≠ pending t1 p) rs'') .
    fst r ≠ p using 9 by force
  show ∃ rs . dstate t2 = dstate t1 * rs ∧ (∀ r ∈ set rs . fst r ≠ p)
    using 8 10 11 by metis
qed
qed
next
case Lin2
assume 1:P17 (s1,s2)
{ fix p
  assume 2:status s1 p ≠ Aborted
  have ∃ rs' . dstate t2 = dstate s2 * rs'
    ∧ (∀ r ∈ set rs' . fst r ≠ p)
  proof -
    obtain rs' where 5:dstate t2 = dstate s2 * rs'
      and 6:rs' ∈ pendingSeqs s2 using Lin2(1) by force
    have 7:∀ r ∈ set rs' . fst r ≠ p
    proof (rule ballI)
      fix r
      assume r ∈ set rs'
      with 6 have r ∈ pendingReqs s2 by (auto simp add:pendingSeqs-def)
      moreover
      have P2 (s1,s2) using reach P2-invariant
        by (metis invariant-def)
      moreover
      have status s2 p = Sleep
    proof -
      have P6 (s1,s2) using reach P6-invariant
        by (metis invariant-def)
      thus ?thesis using 2 Lin2(1) by force
    qed
    ultimately show fst r ≠ p by (auto simp add:pendingReqs-def)
  qed
  show ?thesis using 5 7 by force
}
qed }
note a = this
show P17 (t1,t2)
proof auto
  fix p
  assume 2:initialized t2 and 3:status t1 p = Ready
  obtain rs where dstate s2 = dstate s1 * rs
    and ∀ r ∈ set rs . fst r ≠ p
  proof -
    have initialized s2 and status s1 p = Ready
      using Lin2(1) 2 3 by auto
    thus ?thesis using that 1 by fastforce
  qed

```

```

moreover
obtain  $rs'$  where  $dstate\ t2 = dstate\ s2 \star rs'$ 
  and  $\forall r \in set\ rs' . fst\ r \neq p$  using a 3  $Lin2(1)$ 
  by (metis  $SLin-status.distinct(11)$ )
ultimately show  $\exists rs . dstate\ t2 = dstate\ t1 \star rs$ 
   $\wedge (\forall r \in set\ rs . fst\ r \neq p)$  using  $Lin2(1)$ 
  by auto (metis  $UnE\ exec-append\ set-append$ )
next
fix  $p$ 
assume 2:initialized  $t2$  and 3:status  $t1\ p = Pending$ 
  and 4:contains ( $dstate\ t1$ ) (pending  $t1\ p$ )
obtain  $rs$  where  $dstate\ s2 = dstate\ s1 \star rs$ 
  and  $\forall r \in set\ rs . fst\ r \neq p$ 
proof –
  have initialized  $s2$  and status  $s1\ p = Pending$ 
     $\wedge$  contains ( $dstate\ s1$ ) (pending  $s1\ p$ )
    using  $Lin2(1)$  2 3 4 by auto
  thus ?thesis using that 1 by fastforce
qed
moreover
obtain  $rs'$  where  $dstate\ t2 = dstate\ s2 \star rs'$ 
  and  $\forall r \in set\ rs' . fst\ r \neq p$  using a 3  $Lin2(1)$ 
  by (metis  $SLin-status.distinct(9)$ )
ultimately show  $\exists rs . dstate\ t2 = dstate\ t1 \star rs$ 
   $\wedge (\forall r \in set\ rs . fst\ r \neq p)$  using  $Lin2(1)$ 
  by auto (metis  $UnE\ exec-append\ set-append$ )
next
fix  $p$ 
assume 2:initialized  $t2$  and 3:status  $t1\ p = Pending$ 
  and 4: $\neg$  contains ( $dstate\ t1$ ) (pending  $t1\ p$ )
obtain  $rs$  where  $dstate\ s2 = dstate\ s1 \star rs$ 
  and  $\forall r \in set\ rs . fst\ r = p \longrightarrow r = pending\ s1\ p$ 
proof –
  have initialized  $s2$  and status  $s1\ p = Pending$ 
     $\wedge$   $\neg$  contains ( $dstate\ s1$ ) (pending  $s1\ p$ )
    using  $Lin2(1)$  2 3 4 by auto
  thus ?thesis using that 1 by fastforce
qed
moreover
obtain  $rs'$  where  $dstate\ t2 = dstate\ s2 \star rs'$ 
  and  $\forall r \in set\ rs' . fst\ r \neq p$  using a 3  $Lin2(1)$ 
  by (metis  $SLin-status.distinct(9)$ )
ultimately show  $\exists rs . dstate\ t2 = dstate\ t1 \star rs$ 
   $\wedge (\forall r \in set\ rs . fst\ r = p \longrightarrow r = pending\ t1\ p)$ 
  using  $Lin2(1)$ 
  by auto (metis  $UnE\ exec-append\ set-append$ )
qed
next
case Reco2

```

```

assume 0:P17 (s1,s2)
obtain rs' where 1:dstate t2 = dstate t1 * rs'
and 2:set rs' ⊆ pendingReqs s1 ∪ pendingReqs s2
proof -
obtain ivs rs where 1:ivs ⊆ initVals s2 and 2:ivs ≠ {}
and 3:dstate t2 = ∏ ivs * rs and 4:rs ∈ pendingSeqs s2
using Reco2(1) by (simp add:safeInits-def initSets-def, force)
obtain rs'' where set rs'' ⊆ pendingReqs s1
and ∏ ivs = dstate s1 * rs''
proof -
have P8a (s1,s2) using reach P8a-invariant
by (metis invariant-def)
thus ?thesis using that using 1 2
by (auto simp add:initSets-def pendingSeqs-def)
qed
hence dstate t2 = dstate t1 * (rs''@rs)
∧ set rs'' ⊆ pendingReqs s1
∧ set rs ⊆ pendingReqs s2
using 3 4 Reco2(1) 4
by (metis exec-append mem-Collect-eq pendingSeqs-def)
thus ?thesis using that by force
qed
{ fix p r
assume 1:r ∈ pendingReqs s2
and 2:status s1 p ≠ Aborted
have fst r ≠ p
proof -
have P2 (s1,s2) using reach P2-invariant
by (metis invariant-def)
moreover
have P6 (s1,s2) using reach P6-invariant
by (metis invariant-def)
ultimately show ?thesis using 1 2 Reco2(1)
by (simp add:pendingReqs-def)
(metis SLin-status.distinct(1,5))
qed }
note 3 = this
{ fix r p
assume 1:r ∈ pendingReqs s1 and 2:fst r = p
and 3:status s1 p = Pending
have r = pending s1 p
proof -
have P1 (s1,s2) using reach P1-invariant
by (metis invariant-def)
thus ?thesis using 1 2 3
by (auto simp add:pendingReqs-def)
qed }
note 10 = this
show P17 (t1,t2)

```

```

proof (auto)
  fix  $p$ 
  assume  $4: \text{status } t1 \ p = \text{Ready}$ 
  show  $\exists rs . \text{dstate } t2 = \text{dstate } t1 \star rs$ 
     $\wedge (\forall r \in \text{set } rs . \text{fst } r \neq p)$ 
  proof –
    { fix  $r$ 
      assume  $5: r \in \text{pendingReqs } s1$ 
      have  $\text{fst } r \neq p$ 
      proof –
        have  $P1 (s1, s2)$  using reach P1-invariant
          by (metis invariant-def)
        with  $4 \ 5 \ \text{Reco2}(1)$  show ?thesis
          by (auto simp add: pendingReqs-def)
        qed }
      moreover
      have  $\bigwedge r . r \in \text{pendingReqs } s2 \implies \text{fst } r \neq p$ 
        using  $3 \ 4 \ \text{Reco2}(1)$  by auto
      ultimately show ?thesis using  $1 \ 2$  by blast
    }
  qed
next
  fix  $p$ 
  assume  $4: \text{status } t1 \ p = \text{Pending}$ 
  and  $5: \text{contains } (\text{dstate } t1) (\text{pending } t1 \ p)$ 
  show  $\exists rs . \text{dstate } t2 = \text{dstate } t1 \star rs$ 
     $\wedge (\forall r \in \text{set } rs . \text{fst } r \neq p)$ 
  proof –
    let  $?rs = \text{filter } (\lambda r . r \neq \text{pending } t1 \ p) \ rs'$ 
    have  $\text{dstate } t2 = \text{dstate } t1 \star ?rs$ 
      using  $5 \ 1 \ \text{idem-star}$  by metis
    moreover
    { fix  $r$ 
      assume  $r \in \text{set } ?rs$ 
      have  $\text{fst } r \neq p$ 
      proof –
        { fix  $r$ 
          assume  $6: r \in \text{set } rs'$  and  $7: \text{fst } r = p$ 
          have  $r = \text{pending } s1 \ p$ 
          proof –
            have  $\bigwedge r . r \in \text{pendingReqs } s2 \implies \text{fst } r \neq p$ 
              using  $3 \ 4 \ \text{Reco2}(1)$  by auto
            moreover
            have  $\bigwedge r . r \in \text{pendingReqs } s1 \implies \text{fst } r = p$ 
               $\implies r = \text{pending } s1 \ p$ 
              using  $10 \ 4 \ \text{Reco2}(1)$  by auto
            ultimately show ?thesis using  $2 \ 6 \ 7$ 
              by (metis (lifting, no-types) UnE subsetD)
            qed }
          thus ?thesis using  $\langle r \in \text{set } ?rs \rangle \ \text{Reco2}(1)$  by fastforce
        }
      }
    }
  
```

```

      qed }
    ultimately show ?thesis by blast
  qed
next
fix p
assume 4:status t1 p = Pending
  and 5: $\neg$  contains (dstate t1) (pending t1 p)
show  $\exists$  rs . dstate t2 = dstate t1  $\star$  rs
 $\wedge$  ( $\forall$  r  $\in$  set rs . fst r = p  $\longrightarrow$  r = pending t1 p)
proof -
  have  $\bigwedge$  r . r  $\in$  pendingReqs s2  $\implies$  fst r  $\neq$  p
    using 3 4 Reco2(1) by auto
  moreover
  have  $\bigwedge$  r . r  $\in$  pendingReqs s1  $\implies$  fst r = p
     $\implies$  r = pending s1 p
    using 10 4 Reco2(1) by auto
  ultimately show ?thesis using 1 2 Reco2(1)
    by (metis (lifting, no-types) UnE rev-subsetD)
  qed
qed
qed
qed

```

```

lemma P21-invariant:
shows invariant (composition) P21
proof (rule invariantI, simp-all only:split-paired-all)
  fix s1 s2
  assume (s1,s2)  $\in$  ioa.start (composition)
  thus P21 (s1,s2) using ids by (auto simp add:comp-simps)
next
fix s1 s2 t1 t2 a
assume hyp: P21 (s1,s2) and trans:(s1,s2) -a-composition $\longrightarrow$  (t1,t2)
and reach: reachable (composition) (s1,s2)
show P21 (t1,t2)
proof (cases initialized t2)
  case True
  moreover
  have P12:P12 (t1,t2) using P12-invariant reach trans
    by (metis invariant-def reachable-n)
  moreover
  have P11:P11 (t1,t2) using P11-invariant reach trans
    by (metis IOA.invariant-def reachable-n)
  moreover
  have P20:P20 (t1,t2) using P20-invariant reach trans
    by (metis IOA.invariant-def reachable-n)
  ultimately show P21 (t1,t2) by simp
    (metis pre-RDR.trans)
  case False
next
case False

```

```

show P21 (t1,t2) using trans
proof (cases rule:trans-elim)
  case (Switch2 p c av)
  obtain av where abortVals t2 = abortVals s2 ∪ {av}
  and  $\sqcap$ (abortVals s1)  $\preceq$  av
  proof -
  obtain ivs rs where 1:abortVals t2 = abortVals s2 ∪ { $\sqcap$  ivs  $\star$  rs}
  and 2:ivs  $\subseteq$  initVals s2 and 3:ivs  $\neq$  {}
  using False Switch2(1) by (auto simp add:safeAborts-def
  uninitAborts-def initSets-def)
  have 4: $\sqcap$ (abortVals s1)  $\preceq$   $\sqcap$  ivs
  proof -
  have P11 (s1,s2) using reach P11-invariant
  by (metis invariant-def)
  moreover
  have P13 (s1,s2) using reach P13-invariant
  by (metis invariant-def)
  ultimately show ?thesis
  using 2 3 by (simp add: subset-imp)
  qed
  show ?thesis using that 1 4 by simp
  (metis coboundedI2 less-eq-def orderE)
  qed
with hyp show ?thesis using Switch2(1) by simp
next
case (Switch1 p c av)
show ?thesis
proof (cases abortVals s1 = {})
  case False
  have  $\sqcap$  (abortVals t1)  $\preceq$   $\sqcap$  (abortVals s1)
  proof -
  obtain av where abortVals t1 = abortVals s1 ∪ {av}
  using Switch1(1) by auto
  moreover
  have P13 (s1,s2) using reach P13-invariant
  by (metis invariant-def)
  ultimately show ?thesis using False by simp
  qed
  moreover have abortVals t2 = abortVals s2
  using Switch1(1) by auto
  ultimately show ?thesis using hyp
  by auto (metis coboundedI2 orderE)
next
case True
have abortVals t2 = {}
proof -
  have P19 (s1,s2) using reach P19-invariant
  by (metis invariant-def)
  thus ?thesis using True Switch1(1) by auto

```



```

      qed
      thus ?thesis by auto
    qed
  next
    case (Invoke1 p c)
    thus ?thesis using hyp by simp
  next
    case (Invoke2 p c)
    thus ?thesis using hyp by simp
  next
    case (Response1 p ou)
    thus ?thesis using hyp by simp
  next
    case (Response2 p ou)
    thus ?thesis using hyp by simp
  next
    case Lin1
    thus ?thesis using hyp by auto
  next
    case Lin2
    thus ?thesis using hyp by auto
  next
    case Reco2
    thus ?thesis using hyp by auto
  qed
qed
qed

```

```

lemma P22-invariant:
shows invariant (composition) P22
proof (auto simp only:invariant-def)
  fix s1 s2
  assume 1:reachable (composition) (s1,s2)
  have P9:P9 (s1,s2) using P9-invariant 1
  by (simp add:invariant-def)
  show P22 (s1,s2)
  proof (simp only:P22.simps, rule impI)
    assume initialized s2
    show dstate (f (s1,s2)) = dstate s2
    proof (cases dstate s2 =  $\perp$ )
      case False
      thus ?thesis by auto
    next
      case True
      show dstate (f (s1,s2)) = dstate s2
      proof -
        have dstate s1  $\preceq$  dstate s2
        using ⟨initialized s2⟩ and ⟨P9 (s1,s2)⟩
        by auto
      qed
    qed
  qed

```

```

    hence  $dstate\ s1 = dstate\ s2$  using True
      by (metis antisym bot)
    thus ?thesis by auto
  qed
qed
qed
qed

lemma P23-invariant:
shows invariant (composition) P23
proof (auto simp only:invariant-def)
  fix  $s1\ s2$ 
  assume  $1:reachable\ (composition)\ (s1,s2)$ 
  show P23 (s1,s2)
  proof (simp only:P23.simps, clarify)
    fix  $rs$ 
    assume  $2:\neg initialized\ s2$  and  $3:rs \in pendingSeqs\ s1$ 
    show  $rs \in pendingSeqs\ (f\ (s1,s2))$ 
    proof -
      { fix  $r$ 
        assume  $3:r \in pendingReqs\ s1$ 
        have  $4:status\ s1\ (fst\ r) = Pending \vee status\ s1\ (fst\ r) = Aborted$ 
          and  $5:pending\ s1\ (fst\ r) = r$ 
        proof -
          have  $P1\ (s1,s2)$  using  $1\ P1\text{-invariant}$ 
            by (metis invariant-def)
          thus  $status\ s1\ (fst\ r) = Pending \vee status\ s1\ (fst\ r) = Aborted$ 
            and  $pending\ s1\ (fst\ r) = r$ 
          using  $3$  by (auto simp add:pendingReqs-def)
        qed
        have  $r \in pendingReqs\ (f\ (s1,s2))$  using  $4$ 
        proof
          assume  $status\ s1\ (fst\ r) = Pending$ 
          with  $5$  show ?thesis by (auto simp add:pendingReqs-def)
            (metis SLin-status.distinct(9))
        next
          assume  $6:status\ s1\ (fst\ r) = Aborted$ 
          have  $7:pending\ s1\ (fst\ r) = pending\ s2\ (fst\ r)$ 
             $\wedge status\ s2\ (fst\ r) \in \{Pending, Aborted\}$ 
          proof -
            have  $P7\ (s1,s2)$  using  $1\ P7\text{-invariant}$ 
              by (metis invariant-def)
            thus ?thesis using  $2\ 6$  by auto
          qed
          show ?thesis using  $6\ 5\ 7$  by (simp add:pendingReqs-def, metis)
        qed }
      }
    thus ?thesis using  $3$  by (auto simp only:pendingSeqs-def)
  qed
qed

```

qed

lemma *P26-invariant*:

shows *invariant (composition) P26*

proof (rule *invariantI*, simp-all only:*split-paired-all*)

fix *s1 s2*

assume $(s1,s2) \in \text{ioa.start (composition)}$

thus *P26 (s1,s2) using ids by (auto simp add:comp-simps)*

next

fix *s1 s2 t1 t2 a*

assume *hyp: P26 (s1,s2) and trans:(s1,s2) -a-composition → (t1,t2)*

and *reach:reachable composition (s1,s2)*

show *P26 (t1,t2) using trans and hyp*

proof (cases rule:*trans-elim*)

case *Lin2*

hence $1:\text{dstate } s2 \preceq \text{dstate } t2$

by auto (*metis less-eq-def*)

have $2:t2 = s2(\text{dstate} := \text{dstate } t2)$ and $3:s1 = t1$

using *Lin2(1) by auto*

show *?thesis*

proof (simp, clarify)

fix *p*

assume $4:\text{status } t1 \ p = \text{Aborted}$

and $5:\neg \text{contains (dstate } t2) (\text{pending } t1 \ p)$

have $6:\text{status } s1 \ p = \text{Aborted}$ using *3 4 by auto*

have $7:\text{pending } s1 \ p = \text{pending } t1 \ p$ using *3 by simp*

have $8:\neg \text{contains (dstate } s2) (\text{pending } s1 \ p)$

using *1 5 7*

by simp (*metis contains-star less-eq-def*)

have $11:\text{status } s2 \ p \in \{\text{Pending}, \text{Aborted}\}$

and $9:\text{pending } s1 \ p = \text{pending } s2 \ p$ using *hyp 6 8 by auto*

show $(\text{status } t2 \ p = \text{Pending} \vee \text{status } t2 \ p = \text{Aborted})$

$\wedge \text{pending } t1 \ p = \text{pending } t2 \ p$

proof -

from *2* have $\text{pending } s2 = \text{pending } t2$

and $\text{status } s2 = \text{status } t2$ by ((*cases s2, cases t2, auto*)+)

thus *?thesis using 9 3 11 by auto*

qed

qed

next

case *Reco2*

show *?thesis*

proof (simp, clarify)

fix *p*

assume $1:\text{status } t1 \ p = \text{Aborted}$

have $2:\text{status } s1 \ p = \text{Aborted}$ and $3:\neg \text{initialized } s2$

using *1 Reco2(1) by auto*

have $4:P7 (s1,s2)$ using *reach P7-invariant*

by (*metis invariant-def*)

```

    have 5:status s2 p ∈ {Pending,Aborted}
    and 6:pending s1 p = pending s2 p using 3 4 2 by auto
    show (status t2 p = Pending ∨ status t2 p = Aborted)
      ∧ pending t1 p = pending t2 p using 5 6 Reco2(1) by auto
  qed
next
  case Lin1
  thus ?thesis using hyp by force
next
  case Response1
  thus ?thesis using hyp by force
next
  case Response2
  thus ?thesis using hyp by force
next
  case Invoke2
  thus ?thesis using hyp by force
next
  case Switch1
  thus ?thesis using hyp by force
next
  case Switch2
  thus ?thesis using hyp by force
next
  case Invoke1
  thus ?thesis using hyp by force
qed
qed

lemma P25-invariant:
shows invariant (composition) P25
proof (auto simp only:invariant-def)
  fix s1 s2
  assume reach:reachable (composition) (s1,s2)
  show P25 (s1,s2)
  proof (simp only:P25.simps, clarify)
    fix ivs
    assume 1:ivs ∈ initSets s2 and 2:initialized s2
      and 3:dstate s2 ≼  $\sqcap$  ivs
    obtain rs' where 4:dstate s2  $\star$  rs' =  $\sqcap$  ivs
    and 5:rs' ∈ pendingSeqs s1 and 6: $\forall$  r ∈ set rs' .  $\neg$  contains (dstate s2) r
    proof -
      have 5:dstate s1 ≼ dstate s2
      proof -
        have P9:P9 (s1,s2) using P9-invariant reach
          by (simp add:invariant-def)
        thus ?thesis using 2 by auto
      qed
    obtain rs where 6: $\sqcap$  ivs = dstate s1  $\star$  rs and 7:rs ∈ pendingSeqs s1

```

```

proof –
  have P8a:P8a (s1,s2) using P8a-invariant reach
    by (simp add:invariant-def)
  thus ?thesis using that 1 by auto
qed
have  $\exists rs' . dstate\ s2 \star rs' = \sqcap\ ivs \wedge rs' \in pendingSeqs\ s1$ 
  using 3 5 6 7 consistency[of dstate s1 dstate s2  $\sqcap\ ivs\ rs$ ]
  by (force simp add:pendingSeqs-def)
with this obtain rs' where  $\sqcap\ ivs = dstate\ s2 \star rs'$ 
  and  $rs' \in pendingSeqs\ s1$  by metis
with this show ?thesis using idem-star2 that
  by (metis mem-Collect-eq pendingSeqs-def subset-trans)
qed
have  $\exists rs' \in pendingSeqs\ (f\ (s1,s2))$ 
proof –
  { fix r
    assume  $r \in set\ rs'$ 
    with this obtain p where  $8:status\ s1\ p = Pending$ 
       $\vee\ status\ s1\ p = Aborted$ 
    and  $9:r = pending\ s1\ p$ 
      using 5 by (auto simp add:pendingReqs-def pendingSeqs-def)
    from 8 have  $r \in pendingReqs\ (f\ (s1,s2))$ 
    proof
      assume  $status\ s1\ p = Pending$ 
      thus ?thesis using 9 by (simp add:pendingReqs-def)
        (metis SLin-status.distinct(9))
    next
      assume  $10:status\ s1\ p = Aborted$ 
      hence  $status\ (f\ (s1,s2))\ p = status\ s2\ p$ 
        and  $pending\ (f\ (s1,s2))\ p = pending\ s2\ p$  by simp-all
      moreover
      have  $status\ s2\ p \in \{Pending, Aborted\} \wedge pending\ s2\ p = pending\ s1\ p$ 
      proof –
        have  $\neg\ contains\ (dstate\ s2)\ r$ 
          using 6  $\langle r \in set\ rs' \rangle$  by simp
        moreover
        have P26 (s1,s2) using reach P26-invariant
          by (metis invariant-def)
        ultimately show ?thesis using 10 9 by force
      qed
      ultimately show ?thesis using 9 by (simp only:pendingReqs-def, force)
    }
  thus ?thesis by (auto simp add:pendingSeqs-def)
qed
show  $\exists rs \in pendingSeqs\ (f\ (s1,s2)) . \sqcap\ ivs = dstate\ s2 \star rs$ 
  using 4 7 by force
qed
qed

```

7.4 Proof of the Idempotence Theorem

theorem *idempotence*:

shows $((composition) =_{<} (ioa\ 0\ id2))$

proof –

have *same-input-sig:inp* $(composition) = inp\ (ioa\ 0\ id2)$

– First we show that both automata have the same input and output signature

using *ids* **by** *auto*

moreover

have *same-output-sig:out* $(composition) = out\ (ioa\ 0\ id2)$

– Then we show that output signatures match

using *ids* **by** *auto*

moreover

have *traces* $(composition) \subseteq traces\ (ioa\ 0\ id2)$

– Finally we show trace inclusion

proof –

have *ext* $(composition) = ext\ (ioa\ 0\ id2)$

– First we show that they have the same external signature

using *same-input-sig* **and** *same-output-sig* **by** *simp*

moreover

have *is-ref-map* $f\ (composition)\ (ioa\ 0\ id2)$

– Then we show that *f-comp* is a refinement mapping

proof (*auto simp only:is-ref-map-def*)

fix *s1 s2*

assume $1:(s1,s2) \in ioa.start\ (composition)$

show $f\ (s1,s2) \in ioa.start\ (ioa\ 0\ id2)$

proof –

have $2:ioa.start\ (ioa\ 0\ id2) = start\ (0::nat)$ **by** *simp*

have $3:ioa.start\ (composition)$

$= start\ (0::nat) \times start\ id1$ **by** *fastforce*

show *?thesis*

using $1\ 2\ 3$ **by** *simp*

qed

next

fix *s1 s2 t1 t2* $:: ('a,'b,'c)SLin-state$ **and** *a* $:: ('a,'b,'c,'d)SLin-action$

assume *reach:reachable* $(composition)\ (s1,s2)$

and *trans*: $(s1,s2) -a-(composition) \longrightarrow (t1,t2)$

define *u* **where** $u = f\ (s1,s2)$

define *u'* **where** $u' = f\ (t1,t2)$

Lemmas and invariants

have *pendingReqs* $s2 \subseteq pendingReqs\ u$

proof –

have *P6* $(s1,s2)$ **using** *reach P6-invariant*

by (*metis invariant-def*)

thus *?thesis*

by (*force simp add:pendingReqs-def u-def*)

qed

note *lem1 = this*

have *initialized u* **by** (*auto simp add:u-def*)
have *P1 (s1,s2) and P1 (t1,t2)* **using** *reach P1-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P6 (s1,s2) and P6 (t1,t2)* **using** *reach P6-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P7 (s1,s2) and P7 (t1,t2)* **using** *reach P7-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P8 (s1,s2) and P8 (t1,t2)* **using** *reach P8-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P9 (s1,s2) and P9 (t1,t2)* **using** *reach P9-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P10 (s1,s2) and P10 (t1,t2)* **using** *reach P10-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P13 (s1,s2) and P13 (t1,t2)* **using** *reach P13-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P15 (s1,s2) and P15 (t1,t2)* **using** *reach P15-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P16 (s1,s2) and P16 (t1,t2)* **using** *reach P16-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P17 (s1,s2) and P17 (t1,t2)* **using** *reach P17-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P19 (s1,s2) and P19 (t1,t2)* **using** *reach P19-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P21 (s1,s2) and P21 (t1,t2)* **using** *reach P21-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P22 (s1,s2) and P22 (t1,t2)* **using** *reach P22-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P25 (s1,s2) and P25 (t1,t2)* **using** *reach P25-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P8a (s1,s2) and P8a (t1,t2)* **using** *reach P8a-invariant*
trans invariant-def by (metis , metis reachable-n)
have *P23 (s1,s2) and P23 (t1,t2)* **using** *reach P23-invariant*
trans invariant-def by (metis , metis reachable-n)

show $\exists e . \text{refines } e (s1,s2) a (t1,t2) (\text{ioa } 0 \text{ id2}) f$

using *trans*

proof (*cases rule:trans-elim*)

case (*Invoke1 i p c*)

let $?e = (u, [(a, u')])$

have *1:is-exec-frag-of (ioa 0 id2) ?e*

proof –

have *1:status s1 p = Ready and 2:t2 = s2*

and *3:t1 = s1 (pending := (pending s1)(p := (p,c)),*

status := (status s1)(p := Pending))

using *Invoke1(1) by auto*

have *4:status u p = Ready using 1 u-def by auto*

have *5:u' = u (pending := (pending u)(p := (p,c)),*

status := (status u)(p := Pending))

using *2 3 u-def u'-def by auto*

```

    have 6:Inv p c u u' using 4 5 by force
    show ?thesis using 6 Invoke1(3) ids by simp
qed
have 2:a ∈ ext (ioa 0 id2) and 3:trace (ioa.asig (ioa 0 id2)) ?e = [a]
    using Invoke1(2,3) ids by (auto simp add:trace-def schedule-def fil-
ter-act-def)
show ?thesis using 1 2 3
    by (simp only:refines-def u-def u'-def)
    (metis fst-conv last-state.simps(2) snd-conv)
next

case (Invoke2 i p c)
let ?e = (u,[(a,u')])
have 1:is-exec-frag-of (ioa 0 id2) ?e
proof -
    have 1:status s2 p = Ready and 2:t1 = s1
    and 3:t2 = s2(|pending := (pending s2)(p := (p,c)),
    status := (status s2)(p := Pending))
    using Invoke2(1) by auto
    have 4:status u p = Ready using 1 u-def ⟨P6 (s1,s2)⟩ by auto
    have 5:u' = u(|pending := (pending u)(p := (p,c)),
    status := (status u)(p := Pending))
    using 2 3 u-def u'-def ⟨P6 (t1,t2)⟩ by fastforce
    have 6:Inv p c u u' using 4 5 by force
    show ?thesis using 6 Invoke2(3) ids by simp
qed
have 2:a ∈ ext (ioa 0 id2)
and 3:trace (ioa.asig (ioa 0 id2)) ?e = [a]
    using Invoke2(2,3) by (auto simp add:trace-def schedule-def filter-act-def)
show ?thesis using 1 2 3
    by (simp only:refines-def u-def u'-def)
    (metis fst-conv last-state.simps(2) snd-conv)
next

case (Response2 i p ou)
let ?e = (u,[(a,u')])
have 1:is-exec-frag-of (ioa 0 id2) ?e
proof -
    have 1:status s1 p = Aborted ∧ status t1 p = Aborted
    proof -
        show ?thesis using ⟨P6 (s1,s2)⟩ ⟨P6 (t1,t2)⟩
        Response2(1) by force
    qed
    have 2:status u p = Pending ∧ initialized u
    using 1 Response2(1) u-def by auto
    have 3:u' = u(|status := (status u)(p := Ready))
    using 1 Response2(1) u-def u'-def
    by (cases u, cases u', auto)
    have 4:ou = γ (dstate u) (pending u p) ∧ contains (dstate u) (pending u

```


p)

```
proof (cases dstate s2 =  $\perp$ )
  case False
  thus ?thesis using 1 Response2(1) u-def by auto
next
  case True
  have dstate s1 = dstate s2
  proof -
    have dstate s1  $\preceq$  dstate s2
    using Response2(1)  $\langle P9 (s1, s2) \rangle$  by auto
    with True show ?thesis by (metis antisym bot)
  qed
  thus ?thesis using 1 Response2(1) u-def by auto
  qed
  show ?thesis using 2 3 4 Response2(3) ids by auto
qed
have 2:a  $\in$  ext (ioa 0 id2)
and 3:trace (ioa.asig (ioa 0 id2)) ?e = [a]
  using Response2(2,3) ids
  by (auto simp add:trace-def schedule-def filter-act-def)
show ?thesis using 1 2 3
  by (simp only:refines-def u-def u'-def)
  (metis fst-conv last-state.simps(2) snd-conv)
next

case (Response1 i p ou)
let ?e = (u, [(a, u')])
have 1:is-exec-frag-of (ioa 0 id2) ?e
proof (cases dstate s2 =  $\perp$ )
  case True
  have 1:status u p = Pending  $\wedge$  initialized u
    using Response1(1) u-def by auto
  have 2:u' = u(status := (status u)(p := Ready))
    using Response1(1) u-def u'-def
    by (cases u, cases u', auto)
  have 3:ou =  $\gamma$  (dstate u) (pending u p)
     $\wedge$  contains (dstate u) (pending u p)
    using Response1(1) True u-def by auto
  show ?thesis using 1 2 3  $\langle$ initialized u $\rangle$  Response1(3) ids by auto
next
  case False
  have 1:status u p = Pending  $\wedge$  initialized u
    using Response1(1) u-def by auto
  have 2:u' = u(status := (status u)(p := Ready))
    using Response1(1) u-def u'-def
    by (cases u, cases u', auto)
  have 3:ou =  $\gamma$  (dstate u) (pending u p)
    and 4:contains (dstate u) (pending u p)
  proof -
```

```

have 2:contains (dstate s1) (pending s1 p)
  using Response1(1) by auto
show contains (dstate u) (pending u p)
proof -
  have 3:dstate s1  $\preceq$  dstate u
  proof -
    have initialized s2 using  $\langle P16 (s1,s2) \rangle$  False
      by auto
    thus ?thesis using  $\langle P9 (s1,s2) \rangle$  u-def False refl by simp
  qed
  have 4:pending s1 p = pending u p
    using u-def Response1(1) by force
  show ?thesis
    using 2 3 4 by (metis contains-star less-eq-def)
  qed
have 4: $\gamma$  (dstate s1) (pending s1 p) =  $\gamma$  (dstate u) (pending u p)
proof -
  have 4:pending s1 p = pending u p
    using u-def Response1(1) by force
  obtain rs where 5:dstate u = dstate s1  $\star$  rs
    and 6: $\forall r \in \text{set } rs . \text{fst } r \neq p$ 
  proof -
    have 7:dstate u = dstate s2 using u-def False by simp
    have 6:status s1 p = Pending
       $\wedge$  contains (dstate s1) (pending s1 p)
      using Response1(1) by force
    have 8:initialized s2 using False  $\langle P16 (s1,s2) \rangle$ 
      by auto
    show ?thesis using that  $\langle P17 (s1,s2) \rangle$  6 8 7 by fastforce
  qed
  have 7:fst (pending s1 p) = p
    using Response1(1)  $\langle P1 (s1,s2) \rangle$  by auto
  show ?thesis using 4 5 6 7 2 idem2-star by auto
  qed
thus ou =  $\gamma$  (dstate u) (pending u p)
  using Response1(1) by simp
qed
thus ?thesis using 1 2 3 Response1(3) ids by auto
qed
have 2:a  $\in$  ext (ioa 0 id2)
and 3:trace (ioa.asig (ioa 0 id2)) ?e = [a]
  using Response1(2,3) ids
  by (auto simp add:trace-def schedule-def filter-act-def)
show ?thesis using 1 2 3
  by (simp only:refines-def u-def u'-def)
  (metis fst-conv last-state.simps(2) snd-conv)
next

case (Reco2)

```

```

let ?e = (u,[(Linearize 0,u')])
have is-exec-frag-of (ioa 0 id2) ?e
proof -
  obtain rs where 1:rs ∈ pendingSeqs u
  and 2:dstate u' = dstate u ★ rs
  and 3:∀ av ∈ abortVals u . dstate u' ≼ av
proof -
  obtain rs where set rs ⊆ pendingReqs s1 ∪ pendingReqs s2
  and dstate t2 = dstate s1 ★ rs
  and ∀ av ∈ abortVals s2 . dstate t2 ≼ av
proof -
  obtain ivs rs where 3:ivs ⊆ initVals s2 and 4:ivs ≠ {}
  and 5:dstate t2 = ⌈ ivs ★ rs and 7:rs ∈ pendingSeqs s2
  and 6:∀ av ∈ abortVals s2 . dstate t2 ≼ av
  using Reco2(1)
  by (auto simp add:safeInits-def initSets-def)
  (metis all-not-in-conv)
  obtain rs' where ⌈ ivs = dstate s1 ★ rs'
  and set rs' ⊆ pendingReqs s1
proof -
  { fix iv
    assume 7:iv ∈ ivs
    have ∃ rs . set rs ⊆ pendingReqs s1
      ∧ iv = dstate s1 ★ rs
      using ⟨P8 (s1,s2)⟩ 7 3 by auto
      (metis mem-Collect-eq pendingSeqs-def rev-subsetD) }
  moreover have finite ivs using ⟨P13 (s1,s2)⟩ 3
    by (metis P13.simps rev-finite-subset)
  ultimately show ?thesis using that glb-common-set 4
    by metis
qed
hence dstate t2 = dstate s1 ★ (rs'@rs)
  ∧ set (rs'@rs) ⊆ pendingReqs s1 ∪ pendingReqs s2 using 5 7
  by (metis (lifting, no-types) Un-commute Un-mono
    exec-append mem-Collect-eq pendingSeqs-def set-append)
  thus ?thesis using that 6 by blast
qed
moreover
have pendingReqs s1 ∪ pendingReqs s2 ⊆ pendingReqs u
proof -
  note ⟨pendingReqs s2 ⊆ pendingReqs u⟩
  moreover
  have pendingReqs s1 ⊆ pendingReqs u
  using Reco2(1) ⟨P7 (s1,s2)⟩
  by (auto simp add:pendingReqs-def u-def)
  ultimately show ?thesis by auto
qed
moreover
have abortVals u = abortVals s2 by (auto simp add:u-def)

```

```

moreover
  have  $dstate\ u = dstate\ s1$  using  $\langle P16\ (s1,s2) \rangle$ 
     $Reco2(1)\ u-def$  by force
moreover
  have  $dstate\ u' = dstate\ t2$ 
    using  $Reco2(1)\ \langle P22\ (t1,t2) \rangle$  by  $(auto\ simp\ add:u'-def)$ 
  ultimately show  $?thesis$  using that
    by  $(auto\ simp\ add:pendingSeqs-def,\ blast)$ 
qed
moreover
  have  $u' = u(dstate := dstate\ u \star rs)$ 
    using  $2\ Reco2(1)\ u-def\ u'-def$  by force
moreover
  note  $\langle initialized\ u \rangle$ 
  ultimately show  $?thesis$  by auto
qed
moreover
  have  $a \notin ext\ (ioa\ 0\ id2)$ 
  and  $trace\ (ioa.asig\ (ioa\ 0\ id2))\ ?e = []$ 
    using  $Reco2(2)\ ids$ 
    by  $(auto\ simp\ add:trace-def\ schedule-def\ filter-act-def)$ 
  ultimately show  $?thesis$ 
    by  $(simp\ only:refines-def\ u-def\ u'-def)$ 
     $(metis\ fst-conv\ last-state.simps(2)\ snd-conv)$ 
next

  case  $(Switch1\ p\ c\ av)$ 
  let  $?e = (u, [])$ 
  have  $is-exec-frag-of\ (ioa\ 0\ id2)\ ?e$  by auto
  moreover
  have  $a \notin ext\ (ioa\ 0\ id2)$ 
  and  $trace\ (ioa.asig\ (ioa\ 0\ id2))\ ?e = []$ 
    using  $Switch1(2)\ ids$ 
    by  $(auto\ simp\ add:trace-def\ schedule-def\ filter-act-def)$ 
  moreover
  have  $u = u'$  using  $Switch1(1)\ u-def\ u'-def$  by auto
  ultimately show  $?thesis$ 
    using  $refines-def[of\ ?e\ (s1,s2)\ a\ (t1,t2)\ ioa\ 0\ id2\ f]$ 
     $u-def\ u'-def$  by  $(metis\ last-state.simps(1)\ fst-conv)$ 
next

  case  $Lin2$ 
  let  $?e = (u, [(Linearize\ 0, u')])$ 
  have  $is-exec-frag-of\ (ioa\ 0\ id2)\ ?e$ 
  proof –
    have  $u' = u(dstate := dstate\ u')$  using  $Lin2(1)$ 
      by  $(auto\ simp\ add:u-def\ u'-def)$ 
    moreover
    note  $\langle initialized\ u \rangle$ 

```

```

moreover
obtain  $rs$  where  $dstate\ u' = dstate\ u \star rs$ 
  and  $rs \in pendingSeqs\ u$ 
  and  $\forall av \in abortVals\ u . dstate\ u' \preceq av$ 
proof –
  obtain  $rs$  where  $1:dstate\ t2 = dstate\ s2 \star rs$ 
    and  $2:rs \in pendingSeqs\ s2$ 
    and  $3:\forall av \in abortVals\ s2 . dstate\ t2 \preceq av$ 
    using  $Lin2(1)$  by force
  have  $4:rs \in pendingSeqs\ u$ 
    using  $2$  and  $\langle pendingReqs\ s2 \subseteq pendingReqs\ u \rangle$ 
    by (metis mem-Collect-eq pendingSeqs-def subset-trans)
  have  $5:dstate\ u' = dstate\ u \star rs$ 
    and  $6:\forall av \in abortVals\ u . dstate\ u' \preceq av$ 
proof –
  have  $7:dstate\ u = dstate\ s2 \wedge dstate\ u' = dstate\ t2$ 
    using  $\langle P22\ (s1,s2) \rangle$  and  $\langle P22\ (t1,t2) \rangle$   $Lin2(1)$ 
    by (auto simp add:u-def u'-def)
  show  $dstate\ u' = dstate\ u \star rs$  using  $7\ 1$  by auto
  show  $\forall av \in abortVals\ u . dstate\ u' \preceq av$ 
proof –
    have  $abortVals\ s2 = abortVals\ u$  by (auto simp add:u-def)
    thus ?thesis using  $7\ 3$  by simp
  qed
qed
show ?thesis using that 4 5 6 by auto
qed
ultimately show ?thesis by auto
qed
moreover
have  $a \notin ext\ (ioa\ 0\ id2)$ 
and  $trace\ (ioa.asig\ (ioa\ 0\ id2))\ ?e = []$ 
  using  $Lin2(2)$  ids
  by (auto simp add:trace-def schedule-def filter-act-def)
ultimately show ?thesis
  by (simp only:refines-def u-def u'-def)
  (metis fst-conv last-state.simps(2) snd-conv)
next

case  $Lin1$ 
have  $u' = u \langle dstate := dstate\ u' \rangle$  using  $Lin1(1)$ 
  by (auto simp add:u-def u'-def)
show ?thesis
proof (cases initialized s2)
  case  $False$ 
  let  $?e = (u, [Linearize\ 0, u'])$ 
  have is-exec-frag-of  $(ioa\ 0\ id2)\ ?e$ 
  proof –
    note  $\langle u' = u \langle dstate := dstate\ u' \rangle \rangle$ 

```

```

moreover
note  $\langle \text{initialized } u \rangle$ 
moreover
obtain  $rs$  where  $dstate\ u' = dstate\ u \star rs$ 
  and  $rs \in pendingSeqs\ u$ 
  and  $\forall av \in abortVals\ u . dstate\ u' \preceq av$ 
proof –
  obtain  $rs$  where  $1:dstate\ t1 = dstate\ s1 \star rs$ 
    and  $2:rs \in pendingSeqs\ s1$ 
    and  $3:\forall av \in abortVals\ s1 . dstate\ t1 \preceq av$ 
    using  $Lin1(1)$  by force
  have  $5:pendingSeqs\ s1 \subseteq pendingSeqs\ u$ 
    using  $False\ \langle P7\ (s1,s2) \rangle$ 
    by (auto simp add:pendingReqs-def pendingSeqs-def u-def)
  have  $6:dstate\ u = dstate\ s1 \wedge dstate\ u' = dstate\ t1$ 
    using  $\langle P16\ (s1,s2) \rangle\ False\ Lin1(1)$ 
    by (auto simp add:u-def u'-def)
  have  $4:\forall av \in abortVals\ u . dstate\ u' \preceq av$ 
proof (cases abortVals u = {})
  case True
    thus ?thesis by auto
  next
  case False
    have  $dstate\ u' = dstate\ t1$  using  $6$  by auto
    moreover have  $abortVals\ u = abortVals\ t2$ 
      using  $Lin1(1)$  by (auto simp add:u-def)
    moreover have  $dstate\ t1 \preceq \sqcap(abortVals\ t1)$ 
proof –
    have  $abortVals\ t1 = abortVals\ s1$  using  $Lin1(1)$  by auto
    moreover have  $abortVals\ t1 \neq \{\}$  using  $False\ \langle P19\ (t1,t2) \rangle$ 
       $Lin1(1)$  by (simp add: u-def)
    ultimately show ?thesis using  $3\ \langle P13\ (t1,t2) \rangle$ 
      by simp (metis boundedI)
    qed
    ultimately show ?thesis using  $\langle P21\ (t1,t2) \rangle\ 3$ 
      by (metis P21.simps coboundedI2 orderE)
    qed
    show ?thesis using  $1\ 2\ 3\ 4\ 5\ 6$  that by auto
  qed
  ultimately show ?thesis by auto
qed
moreover
have  $a \notin ext\ (ioa\ 0\ id2)$ 
and  $trace\ (ioa.asig\ (ioa\ 0\ id2))\ ?e = []$ 
  using  $Lin1(2)$  ids
  by (auto simp add:trace-def schedule-def filter-act-def)
ultimately show ?thesis
  by (simp only:refines-def u-def u'-def)
  (metis fst-conv last-state.simps(2) snd-conv)

```

```

next
  case True
  let ?e = (u,[])
  have is-exec-frag-of (ioa 0 id2) ?e by auto
  moreover
  have a ∉ ext (ioa 0 id2)
  and trace (ioa.asig (ioa 0 id2)) ?e = []
  using Lin1(2) ids
  by (auto simp add:trace-def schedule-def filter-act-def)
  moreover have last-state ?e = u'
  proof -
  have dstate u = dstate s2 ∧ dstate u' = dstate t2
  using ⟨P22 (s1,s2)⟩ and ⟨P22 (t1,t2)⟩ and True and Lin1(1)
  by (auto simp add:u-def u'-def)
  thus ?thesis using Lin1(1) ⟨u' = u(dstate := dstate u')⟩
  by simp
  qed
  ultimately show ?thesis
  using refines-def[of ?e (s1,s2) a (t1,t2) ioa 0 id2 f]
  by (simp only:u-def u'-def, auto)
qed
next
  case (Switch2 p c av)
  let ?e = (u,[(a,u')])
  have 1:is-exec-frag-of (ioa 0 id2) ?e
  proof -
  have 1:u' = u(abortVals := (abortVals u) ∪ {av},
  status := (status u)(p := Aborted))
  and 2:av ∈ safeAborts s2 and 3:status u p = Pending
  and 4:pending u p = (p,c)
  proof -
  have 1:t2 = s2(abortVals := (abortVals s2) ∪ {av},
  status := (status s2)(p := Aborted))
  and 2:av ∈ safeAborts s2 and 3:s1 = t1
  and 4:status s2 p = Pending
  using Switch2(1) by auto
  show 5:status u p = Pending using ⟨P6 (s1,s2)⟩ 4
  by (auto simp add:u-def)
  have 6:status u' p = Aborted using ⟨P6 (t1,t2)⟩ 1
  by (auto simp add:u'-def)
  show pending u p = (p,c) using ⟨P6 (s1,s2)⟩ 4 Switch2(1)
  by (auto simp add:u-def)
  show u' = u(abortVals := (abortVals u) ∪ {av},
  status := (status u)(p := Aborted)) using 1 3 5 6
  by (auto simp add:u-def u'-def)
  show av ∈ safeAborts s2 using 2 by assumption
  qed
  have 5:av ∈ safeAborts u

```

```

proof (cases initialized s2)
  case True
  hence 6:dstate u = dstate s2 using ⟨P22 (s1,s2)⟩
    by (auto simp add:u-def)
  have (∃ rs ∈ pendingSeqs s2 . av = dstate s2 ★ rs)
    ∨ (dstate s2 ≼ av ∧ (∃ ivs ∈ initSets s2 .
      dstate s2 ≼ ∏ ivs ∧ (∃ rs ∈ pendingSeqs s2 . av = ∏ ivs ★ rs)))
  proof –
    have av ∈ initAborts s2
      using 2 and True by (auto simp add:safeAborts-def)
    thus ?thesis by (auto simp add:initAborts-def)
  qed
thus ?thesis
proof
  assume ∃ rs ∈ pendingSeqs s2 . av = dstate s2 ★ rs
  moreover note ⟨initialized u⟩
  ultimately show ?thesis using ⟨pendingReqs s2 ⊆ pendingReqs u⟩ 6
    by (simp add:safeAborts-def initAborts-def)
      (metis less-eq-def mem-Collect-eq pendingSeqs-def
        sup.coboundedI2 sup.orderE)
next
  assume 7:dstate s2 ≼ av ∧ (∃ ivs ∈ initSets s2 .
    dstate s2 ≼ ∏ ivs ∧ (∃ rs ∈ pendingSeqs s2 . av = ∏ ivs ★ rs))
  show ?thesis
  proof –
    have 8:dstate u ≼ av using 7 6 by auto
    obtain ivs rs' where 9:ivs ∈ initSets s2
      and 10:dstate s2 ≼ ∏ ivs
      and 11:rs' ∈ pendingSeqs s2 ∧ av = ∏ ivs ★ rs'
      using 7 by auto
    have 12:dstate u = dstate s2 using True ⟨P22 (s1,s2)⟩
      by (auto simp add:u-def)
    moreover
    obtain rs where rs ∈ pendingSeqs u and ∏ ivs = dstate s2 ★ rs
      using ⟨P25 (s1,s2)⟩ True 9 10 by (auto simp add:u-def)
    ultimately have av = dstate u ★ (rs@rs')
      and rs@rs' ∈ pendingSeqs u
      using 11 by (simp-all add:pendingSeqs-def)
        (metis exec-append, metis lem1 subset-trans)
    thus ?thesis using 8 ⟨initialized u⟩
      by (auto simp add:safeAborts-def initAborts-def)
  qed
qed
next
  case False
  with 2 have 0:av ∈ uninitAborts s2 by (auto simp add:safeAborts-def)
  show ?thesis
  proof –
    obtain ivs rs where 1:ivs ∈ initSets s2

```



```

    and 2:rs ∈ pendingSeqs s2
    and 3:av =  $\prod$  ivs  $\star$  rs
    using 0 by (auto simp add:uninitAborts-def)
  have 4:rs ∈ pendingSeqs u using lem1 2
    by (auto simp add:pendingSeqs-def)
  have 5:dstate u = dstate s1 using False ⟨P10 (s1,s2)⟩
    by (auto simp add:u-def)
  obtain rs' where 6: $\prod$  ivs = dstate s1  $\star$  rs'
    and 7:rs' ∈ pendingSeqs s1
    using 1 ⟨P8a (s1,s2)⟩ by auto
  have 8:rs' ∈ pendingSeqs u using False ⟨P23 (s1,s2)⟩ 7
    by (auto simp add:u-def)
  have 9:av = dstate u  $\star$  (rs'@rs) using 3 5 6
    by (metis exec-append)
  have 10:rs'@rs ∈ pendingSeqs u
    using 4 8 by (auto simp add:pendingSeqs-def)
  show ?thesis using 9 10 ⟨initialized u⟩
    by (auto simp add:safeAborts-def initAborts-def less-eq-def)
  qed
  qed
  show ?thesis using 1 3 4 5 Switch2(2) by auto
  qed
  moreover
  have a ∈ ext (ioa 0 id2)
  and trace (ioa.asig (ioa 0 id2)) ?e = [a]
    using Switch2(2) ids
    by (auto simp add:trace-def schedule-def filter-act-def)
  ultimately show ?thesis
    by (simp only:refines-def u-def u'-def)
    (metis fst-conv last-state.simps(2) snd-conv)
  qed
  qed
  ultimately show ?thesis using ref-map-soundness by blast
  qed
  ultimately show ?thesis by (metis ioa-implements-def)
  qed
end
end

```

8 The Consensus Data Type

```

theory Consensus
imports RDR
begin

```

This theory provides a model for the RDR locale, thus showing that the assumption of the RDR locale are consistent.

```

typedecl proc
typedecl val

locale Consensus
— To avoid name clashes
begin

fun  $\delta::val\ option \Rightarrow (proc \times val) \Rightarrow val\ option$  (infix  $\langle \cdot \rangle$  65) where
   $\delta\ None\ r = Some\ (snd\ r)$ 
|  $\delta\ (Some\ v)\ r = Some\ v$ 

fun  $\gamma::val\ option \Rightarrow (proc \times val) \Rightarrow val$  where
   $\gamma\ None\ r = snd\ r$ 
|  $\gamma\ (Some\ v)\ r = v$ 

interpretation pre-RDR  $\delta\ \gamma\ None$  .
notation exec (infix  $\langle \star \rangle$  65)
notation less-eq (infix  $\langle \preceq \rangle$  50 )
notation None ( $\langle \perp \rangle$ )

lemma single-use:
  fixes  $r\ rs$ 
  shows  $\perp \star ([r]@rs) = Some\ (snd\ r)$ 
proof (induct  $rs$ )
  case Nil
  thus ?case by simp
next
  case (Cons  $r\ rs$ )
  thus ?case by auto
qed

lemma bot:  $\exists\ rs . s = \perp \star rs$ 
proof (cases  $s$ )
  case None
  hence  $s = \perp \star []$  by auto
  thus ?thesis by blast
next
  case (Some  $v$ )
  obtain  $r$  where  $\perp \star [r] = Some\ v$  by force
  thus ?thesis using Some by metis
qed

lemma prec-eq-None-or-equal:
fixes  $s1\ s2$ 
assumes  $s1 \preceq s2$ 
shows  $s1 = None \vee s1 = s2$  using assms single-use
proof —
  { assume  $1:s1 \neq None$  and  $2:s1 \neq s2$ 
    obtain  $r\ rs$  where  $3:s1 = \perp \star ([r]@rs)$  using bot using  $1$ 

```

```

    by (metis append-butlast-last-id pre-RDR.exec.simps(1))
  obtain rs' where 4:s2 = s1 * rs' using assms
  by (auto simp add:less-eq-def)
  have s2 = ⊥ * ([r]@(rs@rs')) using 3 4
  by (metis exec-append)
  hence s1 = s2 using 3
  by (metis single-use)
  with 2 have False by auto }
  thus ?thesis by blast
qed

interpretation RDR δ γ ⊥
proof (unfold-locales)
  fix s r
  assume contains s r
  show s · r = s
  proof -
    obtain rs where s = ⊥ * rs and rs ≠ []
    using ⟨contains s r⟩
    by (auto simp add:contains-def, force)
    thus ?thesis
    by (metis δ.simps(2) rev-exhaust single-use)
  qed
next
  fix s and r r' :: proc × val
  assume 1:fst r ≠ fst r'
  thus γ s r = γ ((s · r) · r') r
  by (metis δ.simps γ.simps not-Some-eq)
next
  fix s1 s2
  assume s1 ≼ s2 ∧ s2 ≼ s1
  thus s1 = s2 by (metis prec-eq-None-or-equal)
next
  fix s1 s2
  show ∃ s . is-glb s s1 s2
  by (simp add:is-glb-def is-lb-def)
  (metis bot pre-RDR.less-eq-def prec-eq-None-or-equal)
next
  fix s
  show ⊥ ≼ s
  by (metis bot pre-RDR.less-eq-def)
next
  fix s1 s2 s3 rs
  assume s1 ≼ s2 and s2 ≼ s3 and s3 = s1 * rs
  thus ∃ rs' rs'' . s2 = s1 * rs' ∧ s3 = s2 * rs''
  ∧ set rs' ⊆ set rs ∧ set rs'' ⊆ set rs
  by (metis Consensus.prec-eq-None-or-equal
  in-set-insert insert-Nil list.distinct(1)
  pre-RDR.exec.simps(1) subsetI)

```

qed

end

end

9 Conclusion

In this document we have defined the SLin I/O-automaton (a shorthand for Speculative Linearizability) and we have proved that the composition of two instances of the SLin I/O-automaton behaves like a single instance of the SLin I/O-automaton. This theorem justifies the compositional proof technique presented in [4].

References

- [1] R. Guerraoui, V. Kuncak, and G. Losa. Speculative linearizability. In J. Vitek, H. Lin, and F. Tip, editors, *PLDI*, pages 55–66. ACM, 2012.
- [2] M. P. Herlihy and J. M. Wing. Linearizability: a correctness condition for concurrent objects. *ACM Trans. Program. Lang. Syst.*, 12(3):463–492, 1990.
- [3] L. Lamport and K. Marzullo. The part-time parliament. *ACM Transactions on Computer Systems*, 16:133–169, 1998.
- [4] G. Losa. *Modularity in the design of robust distributed algorithms*. PhD thesis, EPFL, 2014.
- [5] N. A. Lynch and M. R. Tuttle. An introduction to input/output automata. *CWI Quarterly*, 2:219–246, 1989.